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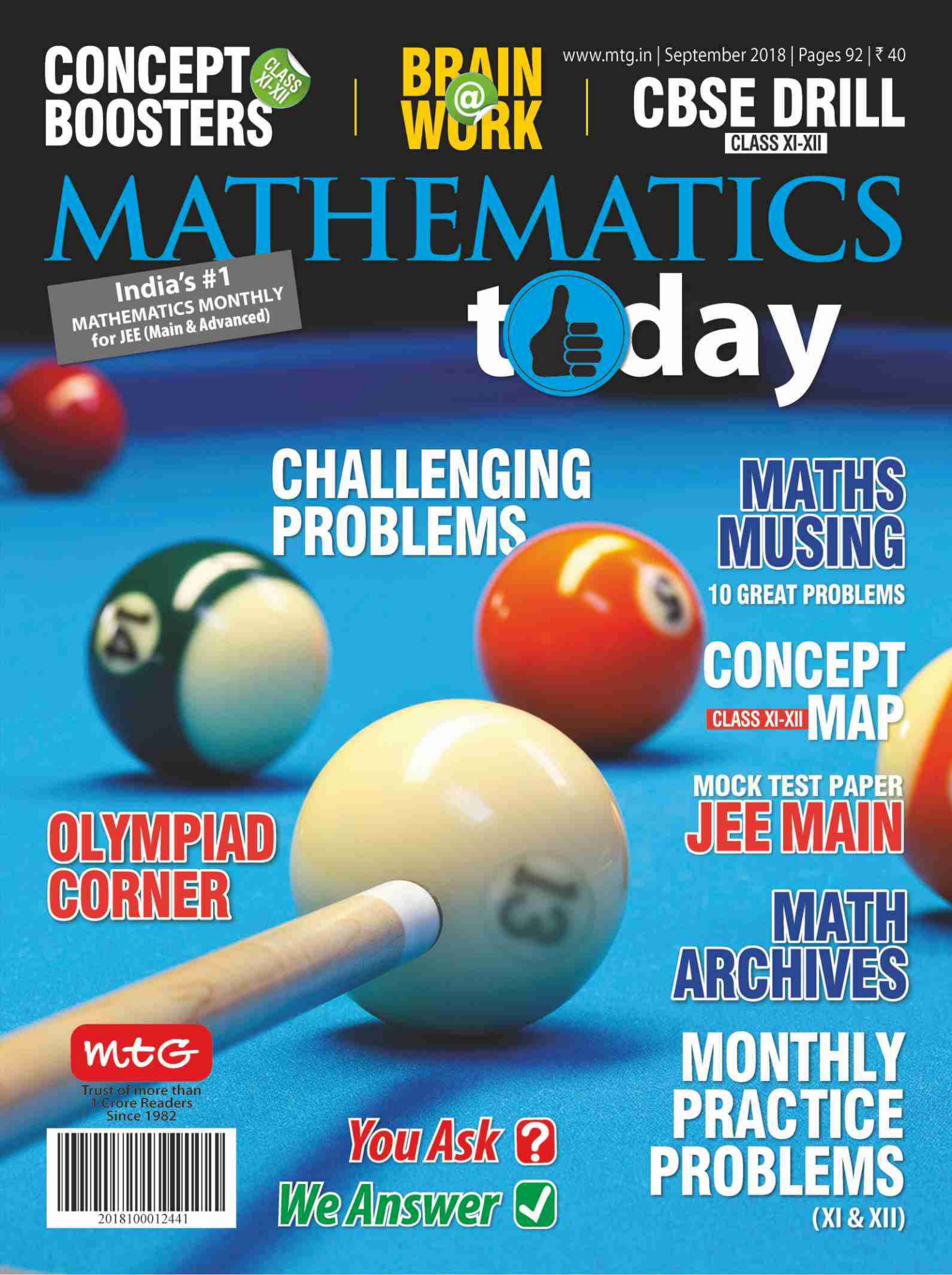

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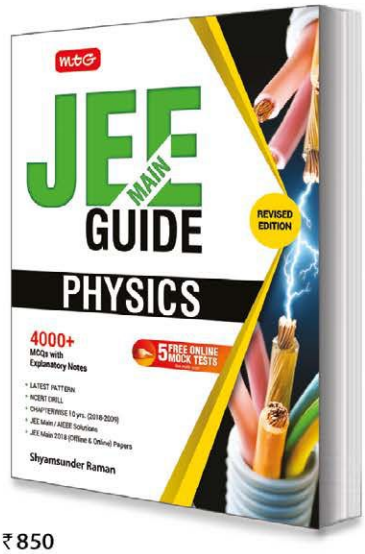

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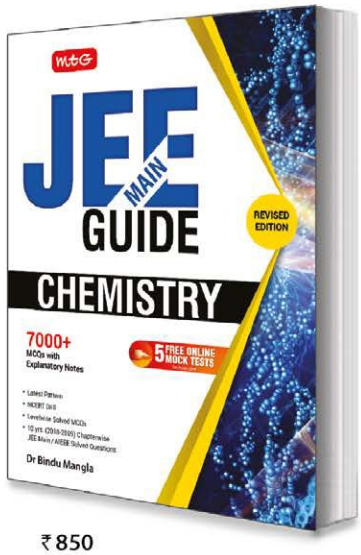

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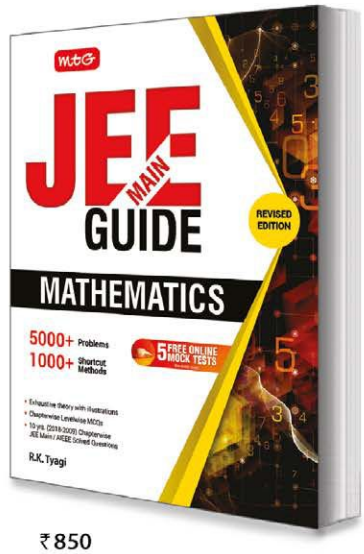

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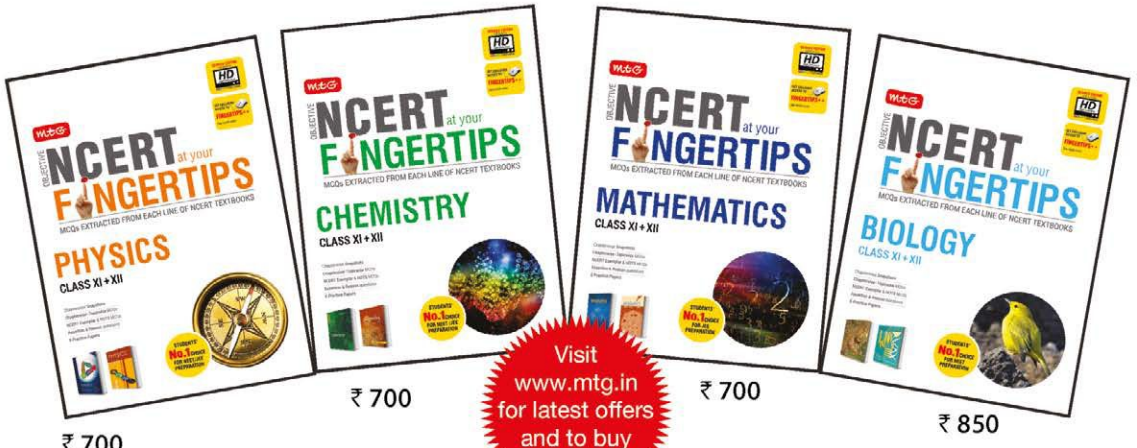

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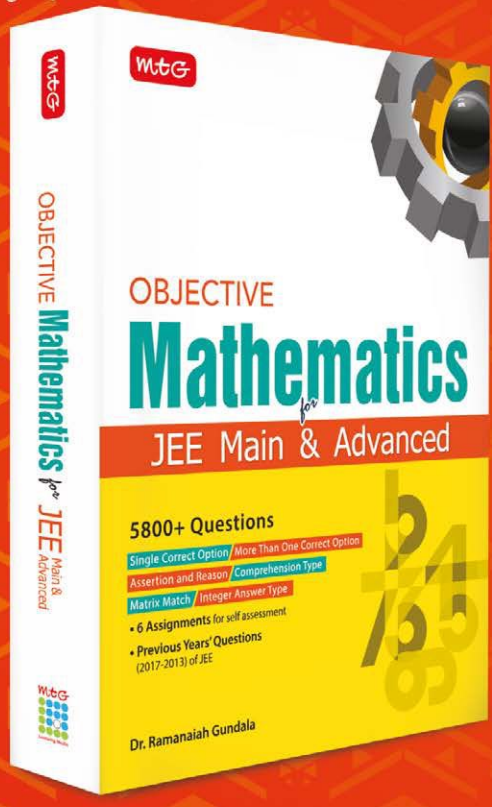
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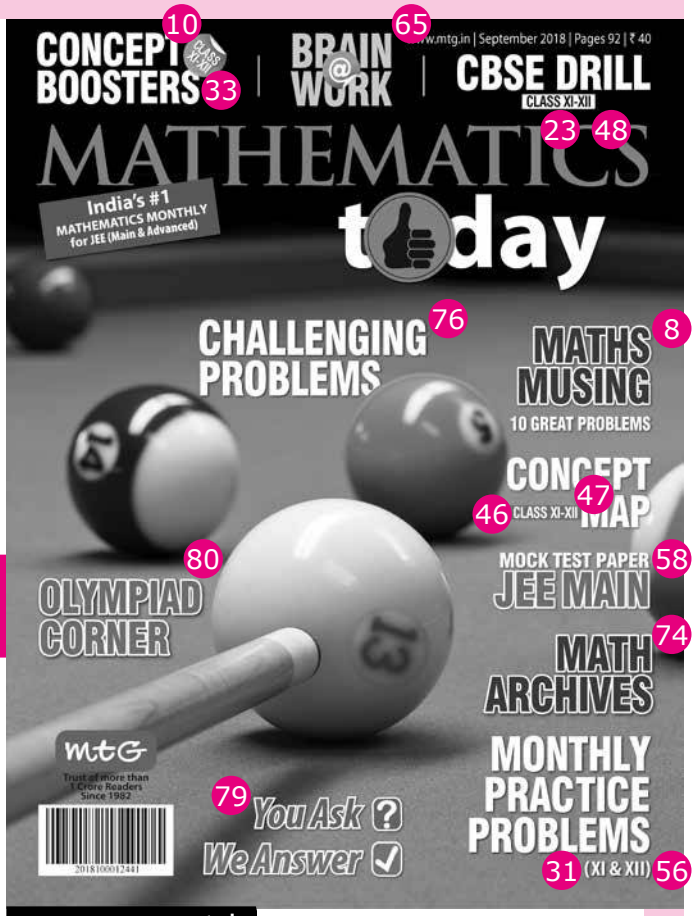
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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 189

JEE MAIN

1. Let $z=\sqrt{3+4i}+\sqrt{-3-4i}$, where square root is taken with positive real part. Then Re(z) is
(a) 3 (b) 4 (c) 2 (d) 1

2. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a<b<c$ and $a+b+c=\frac{3}{2}$. Then a =
(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$
(c) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

3. A man sent 7 letters to his 7 friends. The letters are kept in the addressed envelopes at random. The probability that 3 friends receive correct letters and 4 letters go to wrong destinations, is
(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{3}{32}$ (d) $\frac{5}{64}$

4. Let $f(x)=\int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse function of $f(x)$, then $g''=\lambda g^2$. The value of λ =
(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) 1

5. If the product of the matrices $\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}\cdots\begin{bmatrix}1&n\\0&1\end{bmatrix}=\begin{bmatrix}1&378\\0&1\end{bmatrix}$, then n is equal to
(a) 27 (b) 26 (c) 376 (d) 378

JEE ADVANCED

6. In a triangle ABC, if $\tan A = 2\sin 2C$ and $3\cos A = 2\sin B\sin C$, then C =
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

COMPREHENSION

Consider a triangle ABC formed by the points t_1, t_2, t_3 on the parabola $y^2 = 4ax$. If the orthocentre of the triangle is the focus S of the parabola, then

7. $t_1t_2 + t_2t_3 + t_3t_1$ =
(a) −1 (b) −3 (c) 3 (d) −5

8. $(t_1 - 1)(t_2 - 1)(t_3 - 1)$ =
(a) 1 (b) 2 (c) 3 (d) 4

INTEGER TYPE

9. A circle of centre P touches externally both the circles $x^2 + y^2 - 4x + 3 = 0$ and $x^2 + y^2 - 6y + 5 = 0$. The locus of P is $\frac{3}{4}x^2-3xy+2y^2=\lambda(y-x)$ where λ is

MATRIX MATCH

10. Match the following :

	Column-I	Column-II	
P.	$\lim_{x\to 0}\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}$ =	1.	1
Q.	$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\sec x-\cos x}\,dx$ =	2.	2
R.	Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}$, $x>0$ If $\int_1^2\frac{2}{x}e^{\sin x^2}dx = F(k)-F(1)$, then k =	3.	3
S.	The degree of the differential equation representing the family of curves $y^2 = 2c\left(x+\sqrt{c}\right)$, $c > 0$ being the parameter, is	4.	4

	P	Q	R	S
(a)	3	2	1	4
(b)	4	3	2	1
(c)	1	2	3	4
(d)	1	4	4	3

See Solution Set of Maths Musing 188 on page no. 84

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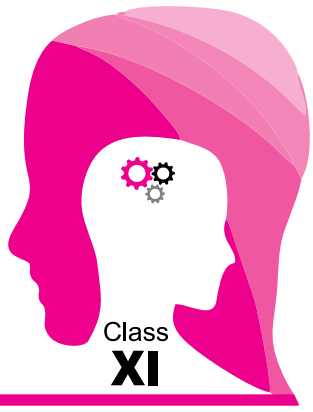
CONCEPT BOOSTERS

Quadratic Equations



This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

An equation of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0$ ($a_0, a_1, \ldots, a_n$ are real coefficients and $a_n \neq 0$) is called a polynomial equation of degree n. This equation is called a linear equation if $n = 1$, quadratic equation if $n = 2$, cubic equation if $n = 3$, bi-quadratic equation if $n = 4$…and so on.

In other words, an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, c are real numbers, is called a quadratic equation. The numbers a, b, c are called the coefficients of the quadratic equation.

ROOTS OF A QUADRATIC EQUATION

A root of the quadratic equation $ax^2 + bx + c = 0$ is a number α (real or complex) such that $a\alpha^2 + b\alpha + c = 0$. The roots of the quadratic equation are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If α and β are two roots of the given quadratic equation then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

DISCRIMINANT OF A QUADRATIC EQUATION

The quantity D ($= b^2 - 4ac$) is known as the discriminant of a quadratic equation.

- The quadratic equation has real and equal roots if and only if $D = 0$ *i.e.* $b^2 - 4ac = 0$.
- The quadratic equation has real and distinct roots if and only if $D > 0$ *i.e.* $b^2 - 4ac > 0$.
- The quadratic equation has complex roots with non-zero imaginary parts if and only if $D < 0$ *i.e.* $b^2 - 4ac < 0$. If $p + iq$ (p and q being real) is a root of the quadratic equation where $i = \sqrt{-1}$, then $p - iq$ is also a root of the quadratic equation.
- The quadratic equation has rational roots if D is a perfect square and a, b, c are rational.
- Quadratic expression $ax^2 + bx + c$ is a perfect square (*i.e.* square of a polynomial of degree one) if $b^2 - 4ac = 0$.

COMMON ROOTS OF TWO QUADRATIC EQUATIONS

- Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α(say).

  Then, $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$.

  Solving, for α^2 and α, we get

$$\frac{\alpha^2}{bf - ce} = \frac{\alpha}{dc - af} = \frac{1}{ae - bd}$$

$$i.e.\ \ \alpha^2 = \frac{bf - ce}{ae - bd} \text{ and } \alpha = \frac{dc - af}{ae - bd}$$

$\Rightarrow \quad (dc - af)^2 = (bf - ce)(ae - bd)$

which is the required condition for the two quadratic equations to have a common root.

To find the common root of two equations, make the coefficient of second degree terms in both the equations equal and then subtract the two equations. The value of x so obtained is the required common root.

Note : Condition for both the roots to be common is $\dfrac{a}{d} = \dfrac{b}{e} = \dfrac{c}{f}$.

If two quadratic equations with real coefficients have an imaginary common root, then both roots will be common for the two equations

---

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

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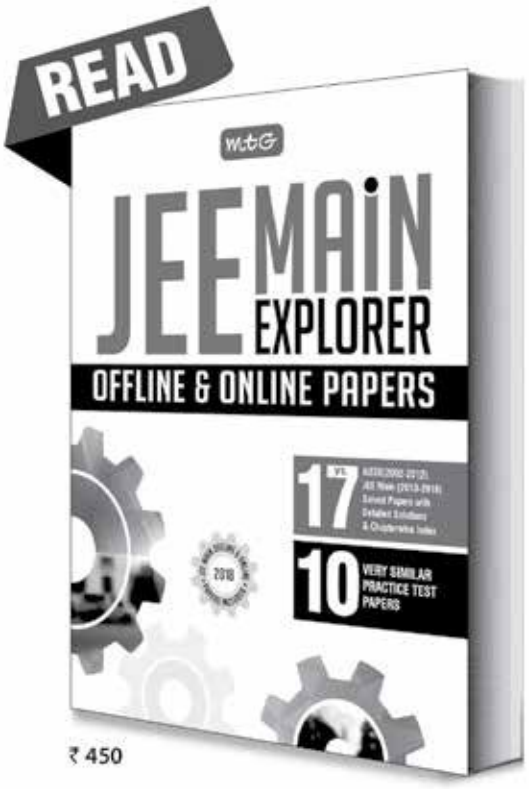

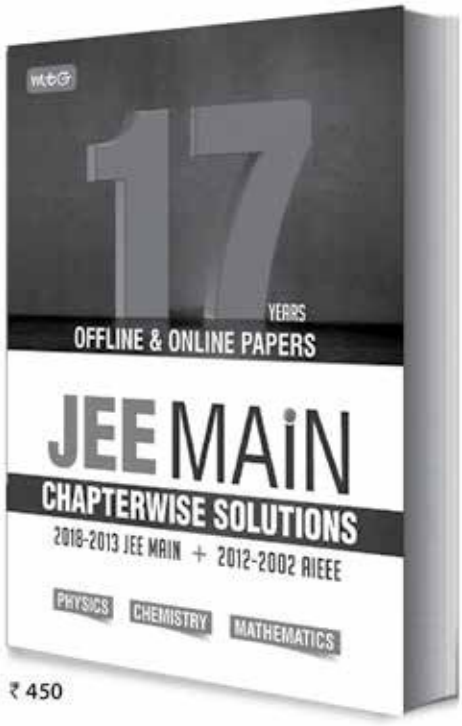

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RANGE OF RATIONAL EXPRESSION OF THE FORM $\frac{ax^2+bx+c}{dx^2+ex+f}$

- To find range, put the given rational expression equal to y and form the quadratic equation in x. Find the discriminant D of the quadratic equation so obtained. Since x is real, therefore, put $D \geq 0$. We get an inequation in y.
- Solve the above inequation for y. The range of y so obtained determines the range attained by the given rational expression.

SIGNS OF A QUADRATIC EXPRESSION

Let $f(x) = ax^2 + bx + c$ where $a, b, c, \in R$ $(a \neq 0)$. Now, $f(x)$ can be rewritten as

$$f(x) = a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\}$$

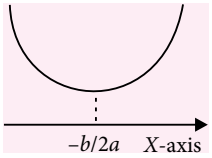
$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \quad \ldots(i)$$

where $D = b^2 - 4ac$ is the discriminant of the quadratic expression. It is clear that $f(x) = ax^2 + bx + c$ will represent a parabola whose axis is parallel to the Y-axis, and vertex is at $A\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$.

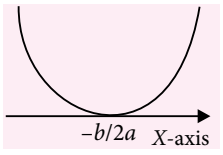
It is also clear that if $a > 0$, the parabola will be open upward and if $a < 0$ the parabola will be open downward and it depends on the sign of $b^2 - 4ac$ that the parabola cuts the X-axis at two points $(b^2 - 4ac > 0)$, touches the X-axis $(b^2 - 4ac = 0)$ or never intersects with the X-axis $(b^2 - 4ac < 0)$.

Case I: If $a > 0$

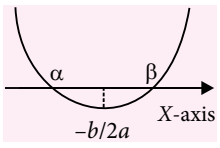
Sub case 1 : $a > 0$ and $b^2 - 4ac < 0$
$\Leftrightarrow f(x) > 0 \ \forall \ x \in R$.
In this case, the parabola always remains open upward and above the X-axis.



Sub case 2 : $a > 0$ and $b^2 - 4ac = 0$
$\Leftrightarrow f(x) \geq 0 \ \forall \ x \in R$.
In this case, the parabola touches the X-axis at one point and remains open upward.



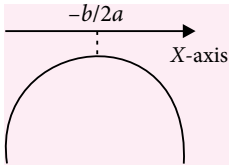
Sub case 3 : $a > 0$ and $b^2 - 4ac > 0$.
Let $f(x) = 0$ has two real roots α and β $(\alpha < \beta)$. Then, $f(x) > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0 \ \forall \ x \in (\alpha, \beta)$.



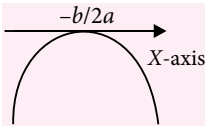
In this case, the parabola cuts the X-axis at two points α and β and remains open upward.

Case II: If $a < 0$

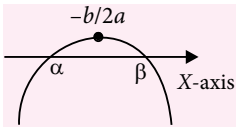
Sub case 1: $a < 0$ and $b^2 - 4ac < 0$
$\Leftrightarrow f(x) < 0 \ \forall \ x \in R$.
In this case, the parabola remains open downward and always below the X-axis.



Sub case 2 : $a < 0$ and $b^2 - 4ac = 0$
$\Leftrightarrow \ f(x) \leq 0 \ \forall \ x \in R$.
In this case, the parabola touches the X-axis and remains open downward.



Sub case 3 : $a < 0$ and $b^2 - 4ac > 0$.
Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$.
Then, $f(x) < 0 \ \forall$ $x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0 \ \forall \ x \in (\alpha, \beta)$.



GREATEST AND LEAST VALUE OF A QUADRATIC EXPRESSION $ax^2 + bx + c$

Case-I : When $a > 0$:
In this case, $ax + bx + c$ has no greatest value but it has least value $\frac{4ac-b^2}{4a}$ at $x = \frac{b}{2a}$.

Case II : When $a < 0$:
If $a < 0$, then $ax^2 + bx + c$ has no least value and it has greatest value $\frac{4ac-b^2}{4a}$ at $x = -\frac{b}{2a}$.

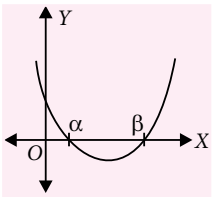
NECESSARY AND SUFFICIENT CONDITIONS ON QUADRATIC EQUATION $ax^2 + bx + c = 0$ $(a > 0)$ FOR ROOTS TO LIE IN A PARTICULAR INTERVAL

Case I: Both the roots are positive *i.e.*, they lie in $(-0, \infty)$, then the sum of the roots as well as the product of the roots must be positive.

Then, $\alpha + \beta = -\frac{b}{a} > 0$

and $\alpha\beta = \frac{c}{a} > 0$

with $b^2 - 4ac \geq 0$

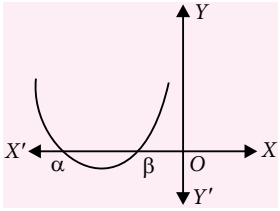


Case II: Both the roots are negative *i.e.* they lie in $(-\infty, 0)$, then the sum of the roots must be negative and the product of the roots must be positive.

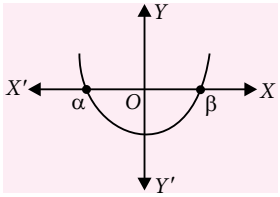
Then, $\alpha+\beta=-\frac{b}{a}<0$

and $\alpha\beta=\frac{c}{a}>0$

with $b^2-4ac\geq 0$



Case III: One root is positive and other is negative *i.e.* origin is lying between the roots. Clearly $f(0)<0$ is the necessary and sufficient condition.



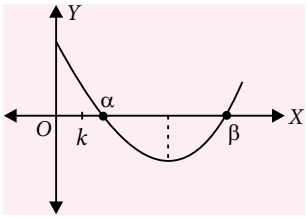
Case IV: Both the roots are greater than a real number k *i.e.*, $k<\alpha<\beta$.

Then the necessary and sufficient conditions are

$D>0$,

$f(k)>0$

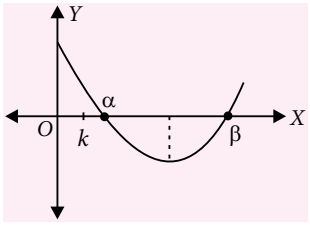
and $k<-\frac{b}{2a}$



Case V: If both the roots are less than a real number k *i.e.*, $\alpha<\beta<k$

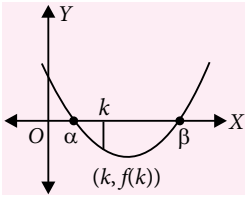
Then, $D>0$,

$f(k)>0$ and $-\frac{b}{2a}<k$

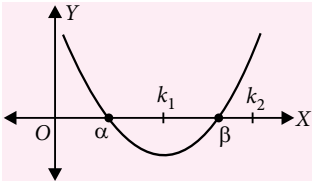


Case VI: A real number k is lying between the roots *i.e.* one root is less than k and other is greater then k. Then,

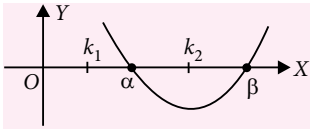
$D>0$ and $f(k)<0$



Case VII: Exactly one root is lying between two real numbers k_1 and k_2



$f(k_1)<0$ and $f(k_2)>0$



$f(k_1)>0$ and $f(k_2)<0$

Hence, the required condition is $f(k_1)\cdot f(k_2)<0$

SOME IMPORTANT POINTS

- $a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+....+a_1x+a_0=0$ ...(i)
where $a_0, a_1...., a_n$ are real coefficients and $a_n\neq 0$
Let $\alpha_1, \alpha_2,...,\alpha_n$ be the roots of equation (i). Then,
$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+.....+a_1x+a_0$
$=a_n(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$
Comparing the coefficients of like powers of x, we get,

$\alpha_1+\alpha_2+\alpha_3+.....+\alpha_n=-\frac{a_{n-1}}{a_n}$

$\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+.....+\alpha_2\alpha_3+.....+\alpha_{n-1}a_n=\frac{a_{n-2}}{a_n}$

$..... \alpha_1\alpha_2.....\alpha_r+.....+\alpha_{n-r+1}\,\alpha_{n-r+2}.....\alpha_n=(-1)^r\frac{a_{n-r}}{a_n}$

$\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\frac{a_0}{a_n}$

e.g. If α, β, γ and δ are the roots of $ax^4+bx^3+cx^2+dx+e=0$. Then,
$\alpha+\beta+\gamma+\delta=-b/a$
$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=c/a$
$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-d/a$
$\alpha\beta\gamma\delta=e/a$
- A polynomial equation of degree n has n roots (real or imaginary).
- If all the coefficients are real then the imaginary roots occur in pair *i.e.* number of complex roots is always even.
- If the degree of a polynomial equation is odd then the number of real roots will also be odd. It follows that at least one of the roots will be real.
- If $p+\sqrt{q}$ is an irrational root of the quadratic equation, then $p-\sqrt{q}$ is other root of the quadratic equation provided that all the coefficients are rational.

- If a, b, c are odd integers, then the roots of quadratic equation can't be rational.
- If $a = 1$ and b, c are integers and the roots of the quadratic equation are rational, then the roots must be integers.
- A quadratic equation, whose roots are α and β can be written as $(x - \alpha)(x - \beta) = 0$ *i.e.*, $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.
- To obtain an equation whose roots are negative of the roots of a given equation, replace x by $-x$.
- If $a + b + c = 0$ and a, b, c are rational, then roots of equation $ax^2 + bx + c = 0$ are rational.
- If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$, then α is a root of the equation $f(x) = 0$.
- Let $f(x) = 0$ be a polynomial equation and p and q are two real numbers, then $f(x) = 0$ will have at least one real root or an odd number of roots between p and q if $f(p)$ and $f(q)$ are of opposite sign. But if $f(p)$ and $f(q)$ are of same signs, then either $f(x)=0$ has no real roots or an even number of roots between p and q.
- If α is repeated root repeating r times of a polynomial equation $f(x) = 0$ of degree n *i.e.* $f(x) = (x - \alpha)^r g(x)$, where $g(x)$ is a polynomial of degree $n - r$ and $g(\alpha) \neq 0$, then $f(\alpha) = f'(\alpha) = \ldots = f^{(r-1)}(\alpha) = 0$ and $f^r(\alpha) \neq 0$.
- The cubic function $f(x) = ax^3 + bx^2 + cx + d$, where $x \in R$, $a > 0$ has the following properties:
  (i) As $x \to \infty$, $y \to \infty$ because the x^3 term is positive and will dominate the remaining terms when x is large.
  (ii) If $f'(x) = 3ax^2 + 2bx + c = 0$ has two real distinct roots p and q (say) where $p > q$, $f(p)$ is a minimum value of $f(x)$ and $f(q)$ is maximum value of $f(x)$.
  (iii) If $f'(x) = 0$ has two real coincident roots, r (say) then r is a point of inflexion for $f(x) = 0$.
  (iv) If $f'(x) = 0$ has no real roots, then the graph of f has no stationary points.
  (v) If $f''(x) = 6ax + 2b$ and $f'(x) = 0, f'''(x) \neq 0$ when $x = -\frac{b}{3a}$. This implies that all cubic curves have a point of inflexion.

PROBLEMS

Single Correct Answer Type

1. If $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$, then x is equal to
(a) 2 (b) 3 (c) 6 (d) 5

2. The value of x in the given equation
$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$ is
(a) $\frac{4}{3}$ (b) $\frac{3}{2}$ (c) $\frac{2}{1}$ (d) $\frac{5}{3}$

3. If the roots of the equation
$(m^2 + 1)x^2 + 2amx + a^2 - b^2 = 0$ are equal, then
(a) $a^2 + b^2(m^2 + 1) = 0$ (b) $b^2 + a^2(m^2 + 1) = 0$
(c) $a^2 - b^2(m^2 + 1) = 0$ (d) $b^2 - a^2(m^2 + 1) = 0$

4. If the roots of the given equation
$(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ are real, then
(a) $p \in (-\pi, 0)$ (b) $p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $p \in (0, \pi)$ (d) $p \in (0, 2\pi)$

5. If $a, b, c \in Q$, then roots of the equation
$(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are
(a) rational (b) non-real
(c) irrational (d) equal

6. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is
(a) 15 (b) 9 (c) 7 (d) 8

7. The value of k for which the equation
$(k - 2)x^2 + 8x + k + 4 = 0$ has both real, distinct and negative roots is
(a) 0 (b) 2 (c) 3 (d) -4

8. If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, then the equation whose roots are α^2 and β^2 is
(a) $4x^2 + 7x + 16 = 0$ (b) $4x^2 + 7x + 6 = 0$
(c) $4x^2 + 7x + 1 = 0$ (d) $4x^2 - 7x + 16 = 0$

9. If the roots of the equation $\frac{\alpha}{x - \alpha} + \frac{\beta}{x - \beta} = 1$ be equal in magnitude but opposite in sign, then $\alpha + \beta =$
(a) 0 (b) 1
(c) 2 (d) none of these

10. The quadratic equation in t, such that A.M. of its roots is A and G.M. is G, is
(a) $t^2 - 2At + G^2 = 0$ (b) $t^2 - 2At - G^2 = 0$
(c) $t^2 + 2At + G^2 = 0$ (d) none of these

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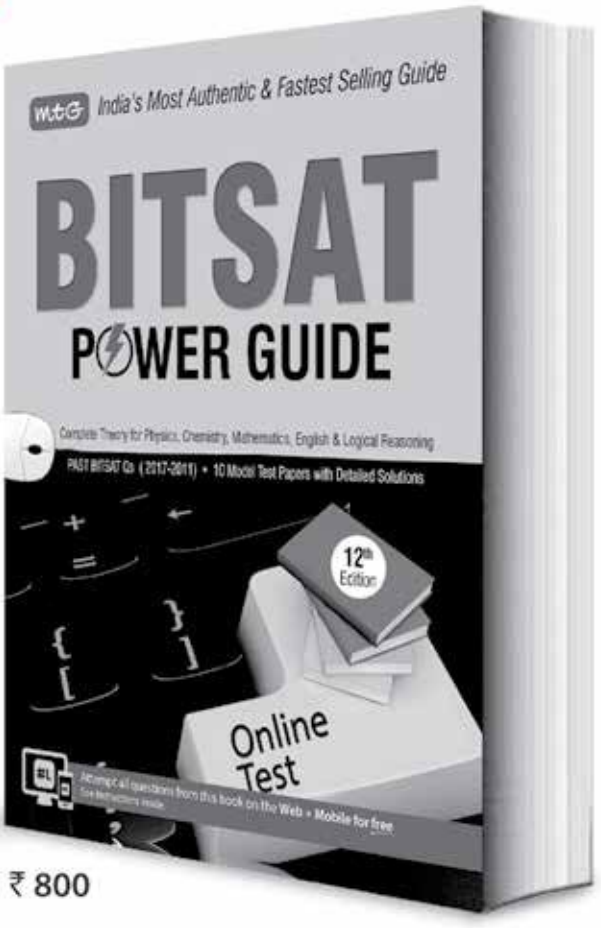
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11. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then a/c, b/a, c/b are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

12. If α and β are roots of $ax^2 + 2bx + c = 0$, then $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}}$ is equal to

(a) $\frac{2b}{ac}$ (b) $\frac{2b}{\sqrt{ac}}$
(c) $-\frac{2b}{\sqrt{ac}}$ (d) $\frac{-b}{\sqrt{2}}$

13. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots will be

(a) $\frac{p^2+q^2}{2}$ (b) $-\frac{(p^2+q^2)}{2}$
(c) $\frac{p^2-q^2}{2}$ (d) $-\frac{(p^2-q^2)}{2}$

14. If the sum of the roots of the equation $x^2 + px + q = 0$ is three times their difference, then which one of the following is true ?

(a) $9p^2 = 2q$ (b) $2q^2 = 9p$
(c) $2p^2 = 9q$ (d) $9q^2 = 2p$

15. If the roots of the equation $ax^2 + bx + c = 0$ are α, β, then the value of $\alpha\beta^2 + \alpha^2\beta + \alpha\beta$ will be

(a) $\frac{c(a-b)}{a^2}$ (b) 0
(c) $-\frac{bc}{a^2}$ (d) none of these

16. The roots of the equation $x^2 + ax + b = 0$ are p and q, then the equation whose roots are p^2q and pq^2 will be

(a) $x^2 + abx + b^3 = 0$ (b) $x^2 - abx + b^3 = 0$
(c) $bx^2 + x + a = 0$ (d) $x^2 + ax + ab = 0$

17. If α and β are the roots of the equation $x^2 - 4x + 1 = 0$, then the value of $\alpha^3 + \beta^3$ is

(a) 76 (b) 52
(c) –52 (d) –76

18. Let α, α^2 be the roots of $x^2 + x + 1 = 0$, then the equation whose roots are α^{31}, α^{62} is

(a) $x^2 - x + 1 = 0$ (b) $x^2 + x - 1 = 0$
(c) $x^2 + x + 1 = 0$ (d) none of these

19. If α and β are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$; a, b, c being different, then $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is

(a) zero (b) positive
(c) negative (d) none of these

20. If the roots of $ax^2 + bx + c = 0$ are α, β and the roots of $Ax^2 + Bx + C = 0$ are $\alpha - k$, $\beta - k$, then $\frac{B^2 - 4AC}{b^2 - 4ac}$ is equal to

(a) 0 (b) 1 (c) $\left(\frac{A}{a}\right)^2$ (d) $\left(\frac{a}{A}\right)^2$

21. If α, β are the roots of $x^2 - 2x + 4 = 0$, then $\alpha^5 + \beta^5$ is equal to

(a) 16 (b) 32
(c) 64 (d) none of these

22. If α, β are the roots of the equation $lx^2 + mx + n = 0$, then the equation whose roots are $\alpha^3\beta$ and $\alpha\beta^3$ is

(a) $l^4x^2 - nl(m^2 - 2nl)x + n^4 = 0$
(b) $l^4x^2 + nl(m^2 - 2nl)x + n^4 = 0$
(c) $l^4x^2 + nl(m^2 - 2nl)x - n^4 = 0$
(d) $l^4x^2 - nl(m^2 + 2nl)x + n^4 = 0$

23. If α, β are the roots of $x^2 - (1 + n^2)x + \frac{1}{2}(1 + n^2 + n^4) = 0$, then the value of $\alpha^2 + \beta^2$ is

(a) $2n$ (b) n^3 (c) n^2 (d) $2n^2$

24. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, then the equation whose roots are $3p - 2q$ and $3q - 2p$ is

(a) $3x^2 - 5x - 100 = 0$ (b) $5x^2 + 3x + 100 = 0$
(c) $3x^2 - 5x + 100 = 0$ (d) $5x^2 - 3x - 100 = 0$

25. If α and β are the roots of $6x^2 - 6x + 1 = 0$, then the value of

$\frac{1}{2}[a + b\alpha + c\alpha^2 + d\alpha^3] + \frac{1}{2}[a + b\beta + c\beta^2 + d\beta^3]$ is

(a) $\frac{1}{4}(a + b + c + d)$ (b) $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$
(c) $\frac{a}{2} - \frac{b}{2} + \frac{c}{3} - \frac{d}{4}$ (d) none of these

26. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, then the value of $\sin^2(\alpha + \beta)$ is

(a) $\frac{p^2}{p^2 + (1-q)^2}$ (b) $\frac{p^2}{p^2 + q^2}$
(c) $\frac{q^2}{p^2 + (1-q)^2}$ (d) $\frac{p^2}{(p+q)^2}$

27. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2 : 3, then m =

(a) $5\sqrt{10}$ (b) $3\sqrt{10}$

(c) $2\sqrt{10}$ (d) none of these

28. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P., where $\Delta = b^2 - 4ac$, then

(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $cb \neq 0$ (d) $c\Delta = 0$

Multiple Correct Answer Type

29. The value of p for which one root of the equation $x^2 - 30x + p = 0$ is the square of the other, are

(a) 125 (b) −216 (c) −125 (d) 216

30. If the roots of the equation $x^2 - 5x + 16 = 0$ are α, β and the roots of equation $x^2 + px + q = 0$ are $\alpha^2 + \beta^2$, $\frac{\alpha\beta}{2}$, then

(a) $p = 1$ (b) $p = -1$ (c) $q = 56$ (d) $q = -56$

31. Given $|ax^2 + bx + c| \leq |Ax^2 + Bx + C| \ \forall \ x \in R$, $a, b, c, A, B, C \in R$, $d = b^2 - 4ac > 0$ and $D = B^2 - 4AC > 0$. Then which of the following statements are true?

(a) $|a| \leq |A|$ (b) $|d| \leq |D|$

(c) $|a| \geq |A|$ (d) $|d| \geq |D|$

32. If α, β, γ are the roots of the equation $9x^3 - 7x + 6 = 0$, then the equation $x^3 + Ax^2 + Bx + C = 0$ has roots $3\alpha + 2, 3\beta + 2, 3\gamma + 2$, where

(a) $A = 6$ (b) $B = -5$

(c) $C = 24$ (d) $A + B + C = 23$

Comprehension Type

Paragraph for Q. No. 33 and 34

Let $f(x) = ax^2 + bx + c$, $a \neq 0$, $a\ b, c \in I$. Suppose that $f(1) = 0$, $50 < f(7) < 60$ and $70 < f(8) < 80$.

33. The least value of $f(x)$ is

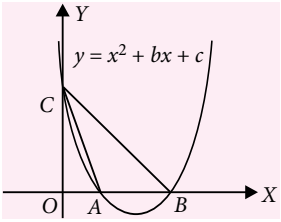
(a) $\frac{3}{4}$ (b) $\frac{9}{2}$ (c) $-\frac{9}{8}$ (d) $\frac{3}{4}$

34. Number of integral values of x for which $f(x) < 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Q. No. 35 and 36

In the given figure ΔOBC is an isosceles right triangle in which AC is median.



35. The equation whose roots are $(\alpha + \beta)$ and $(\alpha - \beta)$, where α, β $(\alpha > \beta)$ are roots of $x^2 + bx + c = 0$, is

(a) $x^2 - 4x + 3 = 0$ (b) $x^2 - 8x + 12 = 0$

(c) $4x^2 - 8x + 3 = 0$ (d) $x^2 - 16x + 48 = 0$

36. Minimum value of the quadratic expression corresponding to the quadratic equation obtained in above question occurs at x =

(a) 8 (b) 1 (c) 4 (d) 2

Matrix–Match Type

37. Match the following.

	Column-I		Column-II
A.	If $x^2 + x - a = 0$ has integral roots and $a \in N$, then a can be equal to	(p)	3
B.	If the equation $ax^2 + 2bx + 4c = 16$ has no real roots and $a + c > b + 4$ then $c/2$ cannot be	(q)	6
C.	If $x^2 + 2bx + 9b - 14 = 0$, has only negative roots, then integral values of b may be	(r)	2

Integer Answer Type

38. Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$. If $P = a^3 + b^3 + c^3$, then the value of $\frac{P}{2006}$ =

39. Let $f(x) = 30 - 2x - x^3$, then find the number of positive integral values of x which satisfies $f(f(f(x))) > f(f(-x))$.

40. If the quadratic equations $ax^2 + 2cx + b = 0$ and $ax^2 + 2bx + c = 0$, $(b \neq c)$ have a common root, then $(a + 4b + 4c)$ is

SOLUTIONS

1. (c) : $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$

$\sqrt{3x^2 - 7x - 30} = (x + 5) - \sqrt{2x^2 - 7x - 5}$

On squaring, we get, $\sqrt{2x^2 - 7x - 5} = 5$

$\Rightarrow \quad 2x^2 - 7x - 30 = 0 \Rightarrow x = 6.$

2. (b) : We have, $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

$\Rightarrow \quad 2^{2x} + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$

$\Rightarrow \quad 2^{2x}\left(1 + \frac{1}{2}\right) = 3^{x-\frac{1}{2}}(1 + 3) \Rightarrow 2^{2x-3} = 3^{x-\frac{3}{2}}$

Taking log on both sides, we get
$(2x - 3)\log 2 = (x - 3/2)\log 3$

$\Rightarrow \quad 2x\log 2 - 3\log 2 = x\log 3 - \frac{3}{2}\log 3$

$\Rightarrow \quad x\log 4 - x\log 3 = 3\log 2 - \frac{3}{2}\log 3$

$\Rightarrow \quad x\log\left(\frac{4}{3}\right) = \log 8 - \log 3\sqrt{3}$

$\Rightarrow \quad \left(\frac{4}{3}\right)^x = \frac{8}{3\sqrt{3}} \Rightarrow \left(\frac{4}{3}\right)^x = \left(\frac{4}{3}\right)^{3/2} \Rightarrow \quad x = \frac{3}{2}.$

3. (c) : Roots are equal, then discriminant = 0
$\Rightarrow \quad 4a^2m^2 = 4(m^2 + 1)(a^2 - b^2)$
$\Rightarrow \quad a^2 - b^2(m^2 + 1) = 0.$

4. (c): Given, $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$
$\because$ Roots are real, its discriminant $D \geq 0$
$\Rightarrow \quad \cos^2 p - 4(\cos p - 1)\sin p \geq 0$
$\Rightarrow \quad \cos^2 p - 4\cos p \sin p + 4\sin p \geq 0$
$\Rightarrow \quad (\cos p - 2\sin p)^2 - 4\sin^2 p + 4\sin p \geq 0$
$\Rightarrow \quad (\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \geq 0$ ...(i)
Now, $(1 - \sin p) \geq 0$ for all real p, $\sin p > 0$ for $0 < p < \pi$.
$\therefore \quad 4\sin p(1 - \sin p) \geq 0$ when $p \in (0, \pi)$

5. (a): Here $(b + c - 2a) + (c + a - 2b) + (a + b - 2c) = 0$
Therefore, the roots are rational.

6. (c) : For real roots, discriminant ≥ 0
$\Rightarrow \quad q^2 - 4p \geq 0 \Rightarrow q^2 \geq 4p$
For $p = 1, q^2 \geq 4 \Rightarrow q = 2, 3, 4$
$p = 2, q^2 \geq 8 \Rightarrow q = 3, 4$
$p = 3, q^2 \geq 12 \Rightarrow q = 4$
$p = 4, q^2 \geq 16 \Rightarrow q = 4$
$\therefore$ Total seven solutions are possible.

7. (c) : From options, put $k = 3$, we get
$x^2 + 8x + 7 = 0$
$\Rightarrow \quad (x + 1)(x + 7) = 0 \Rightarrow x = -1, -7.$

8. (a) : We have, $\alpha + \beta = \frac{3}{2}$ and $\alpha\beta = 2$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} - 4 = -\frac{7}{4}$

$\therefore \quad$ Required equation is $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

$\Rightarrow \quad x^2 + \frac{7}{4}x + 4 = 0 \Rightarrow 4x^2 + 7x + 16 = 0.$

9. (a) : We have, $\frac{\alpha}{x - \alpha} + \frac{\beta}{x - \beta} = 1$
$\Rightarrow \quad x^2 - 2(\alpha + \beta)x + 3\alpha\beta = 0$
Let roots are α' and $-\alpha'$
Now, $\alpha' + (-\alpha') = 0 \Rightarrow 2(\alpha + \beta) = 0 \Rightarrow \alpha + \beta = 0.$

10. (a) : If α, β are the roots, then
$A = \frac{\alpha + \beta}{2} \Rightarrow \alpha + \beta = 2A$ and $G = \sqrt{\alpha\beta} \Rightarrow \alpha\beta = G^2$
The required equation is $t^2 - 2At + G^2 = 0$.

11. (c) : As given, if α, β be the roots of the quadratic equation, then $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$

$\Rightarrow \quad -\frac{b}{a} = \frac{(b^2/a^2) - (2c/a)}{(c^2/a^2)} = \frac{b^2 - 2ac}{c^2}$

$\Rightarrow \quad \frac{2a}{c} = \frac{b^2}{c^2} + \frac{b}{a} = \frac{(ab^2 + bc^2)}{ac^2}$

$\Rightarrow \quad 2a^2c = ab^2 + bc^2 \Rightarrow \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$

$\Rightarrow \quad \frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P. $\Rightarrow \quad \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

12. (c) : Given equation is $ax^2 + 2bx + c = 0$.
So, $\alpha + \beta = -\frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now, $\sqrt{\left(\frac{\alpha}{\beta}\right)} + \sqrt{\left(\frac{\beta}{\alpha}\right)} = \frac{\alpha + \beta}{\sqrt{\alpha\beta}} = \frac{-2b/a}{\sqrt{c/a}} = -\frac{2b}{\sqrt{ac}}$

13. (b) : Given equation can be written as
$x^2 + x(p + q - 2r) + pq - pr - qr = 0$ ...(i)
Roots of (i) are α and $-\alpha$, then we have
Product of roots $= -\alpha^2 = pq - pr - qr = pq - r(p + q)$ ...(ii)

Sum of roots $= 0 \Rightarrow p + q - 2r = 0 \Rightarrow r = \frac{p + q}{2}$ ...(iii)
From (ii) and (iii), we get

$$-\alpha^2 = pq - \frac{p + q}{2}(p + q) = -\frac{1}{2}\{(p + q)^2 - 2pq\}$$
$$= -\frac{(p^2 + q^2)}{2}$$

14. (c) : Let α, β are roots of $x^2 + px + q = 0$
So, $\alpha + \beta = -p$ and $\alpha\beta = q$

Given that $(\alpha + \beta) = 3(\alpha - \beta) \Rightarrow \alpha - \beta = \frac{-p}{3}$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \quad \frac{p^2}{9} = p^2 - 4q$ or $2p^2 = 9q.$

15. (a) : Given, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now, $\alpha\beta^2 + \alpha^2\beta + \alpha\beta = \alpha\beta(\beta + \alpha) + \alpha\beta$

$= \alpha\beta(1 + \alpha + \beta) = \frac{c}{a}\left\{1 + \left(-\frac{b}{a}\right)\right\} = \frac{c(a - b)}{a^2}.$

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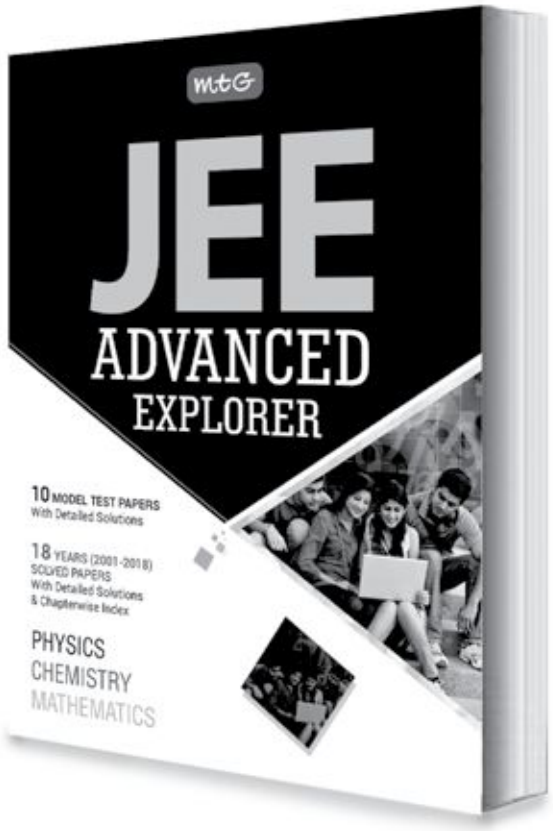

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16. (a) : Given equation is $x^2 + ax + b = 0$

$\Rightarrow \quad p + q = -a$ ...(i) and $pq = b$ ...(ii)

We have to find the equation whose roots are p^2q and pq^2.

Now, sum of roots $= p^2q + pq^2 = pq(p + q) = -ab$ [From (i) and (ii)]

and product of roots $= (pq^2) \cdot (qp^2) = (pq)^3 = b^3$

Thus equation is $x^2 + abx + b^3 = 0$.

17. (b) : $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (4)^3 - (3 \times 1 \times 4) = 52$

18. (c) : Give equation is $x^2 + x + 1 = 0$

$\therefore \quad \alpha + \alpha^2 = -1$ ...(i) and $\alpha^3 = 1$ ...(ii)

We have to find the equation whose roots are α^{31} and α^{62}.

Now, $\alpha^{31} + \alpha^{62} = \alpha^{31}(1 + \alpha^{31})$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = \alpha^{30} \cdot \alpha(1 + \alpha^{30} \cdot \alpha)$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = (\alpha^3)^{10} \cdot \alpha\{1 + (\alpha^3)^{10} \cdot \alpha\}$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = \alpha(1 + \alpha)$

$\Rightarrow \quad \alpha^{31} + \alpha^{62} = -1$ [From (i)]

Again, $\alpha^{31} \cdot \alpha^{62} = \alpha^{93} \Rightarrow \alpha^{31} \cdot \alpha^{62} = [\alpha^3]^{31} = 1$

Required equation is $x^2 - (\alpha^{31} + \alpha^{62})x + \alpha^{31} \cdot \alpha^{62} = 0$

$\Rightarrow \quad x^2 + x + 1 = 0$

19. (b) : $\alpha + \beta = -b/a$, $\alpha\beta = c/a$

$(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$

$= 1 + (\alpha + \beta) + (\alpha^2 + \beta^2) + \alpha\beta + \alpha\beta(\alpha + \beta) + \alpha^2\beta^2$

$= 1 + (\alpha + \beta) + (\alpha + \beta)^2 - \alpha\beta + \alpha\beta(\alpha + \beta) + \alpha^2\beta^2$

$$= 1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{c}{a} + \left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) + \frac{c^2}{a^2}$$

$= (a^2 + b^2 + c^2 - ab - bc - ca)/a^2$

$= [(a - b)^2 + (b - c)^2 + (c - a)^2]/2a^2$ which is positive.

20. (c): $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (b^2 - 4ac)/a^2$ ...(i)

Also, $\{(\alpha - k) - (\beta - k)\}^2$

$= \{(\alpha - k) + (\beta - k)\}^2 - 4(\alpha - k)(\beta - k)$

$= (-B/A)^2 - 4(C/A) = (B^2 - 4AC)/A^2$ ...(ii)

From (i) and (ii), $(b^2 - 4ac)/a^2 = (B^2 - 4AC)/A^2$

$$\therefore \quad \frac{B^2 - 4AC}{b^2 - 4ac} = \left(\frac{A}{a}\right)^2$$

21. (b) : Here, $\alpha^2 + \beta^2 = -4$, $\alpha^3 + \beta^3 = -16$

$\therefore \quad (\alpha^2 + \beta^2)(\alpha^3 + \beta^3) = \alpha^5 + \beta^5 + \alpha^2\beta^2(\alpha + \beta)$

$\Rightarrow \quad (-4)(-16) = \alpha^5 + \beta^5 + (16)(2) \Rightarrow \alpha^5 + \beta^5 = 32.$

22. (a) : Taking $l = 1$, $m = -3$, $n = 2 \Rightarrow \alpha = 2$, $\beta = 1$.

23. (c) : $\alpha + \beta = 1 + n^2$; $\alpha\beta = \frac{1}{2}(1 + n^2 + n^4)$

$\therefore \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= (1 + n^2)^2 - 2 \cdot \frac{1}{2}(1 + n^2 + n^4)$$

$= 1 + n^4 + 2n^2 - 1 - n^2 - n^4 \Rightarrow \alpha^2 + \beta^2 = n^2.$

24. (a) : According to question, p and q are roots of $3x^2 - 5x - 2 = 0$

$$\therefore \quad p + q = \frac{5}{3} \text{ and } pq = \frac{-2}{3}.$$

We have to find equation whose roots are $3p - 2q$ and $3q - 2p$.

$\therefore$ Sum of roots $= (3p - 2q) + (3q - 2p) = (p + q) = \frac{5}{3}$

And product of roots $= (3p - 2q)(3q - 2p)$

$= 9pq - 6q^2 - 6p^2 + 4pq = 13pq - 2(3p^2 + 3q^2)$

$$= 13\left(\frac{-2}{3}\right) - 2(5p + 2 + 5q + 2)$$

$$= 13\left(\frac{-2}{3}\right) - 2\left[5\left(\frac{5}{3}\right) + 4\right] = \frac{-26}{3} - 2\left[\frac{25}{3} + 4\right] = \frac{-100}{3}$$

Hence, equation is $3x^2 - 5x - 100 = 0$.

25. (b) : Since α, β are the roots of the equation $6x^2 - 6x + 1 = 0$

$\therefore \quad \alpha + \beta = 1$, $\alpha\beta = 1/6$

Now, $\frac{1}{2}[a + b\alpha + c\alpha^2 + d\alpha^3] + \frac{1}{2}[a + b\beta + c\beta^2 + d\beta^3]$

$$= a + \frac{1}{2}b(\alpha + \beta) + \frac{1}{2}c(\alpha^2 + \beta^2) + \frac{1}{2}d(\alpha^3 + \beta^3)$$

$$= a + \frac{1}{2}b + \frac{1}{2}c[(\alpha + \beta)^2 - 2\alpha\beta] + \frac{1}{2}d[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]$$

$$= a + \frac{b}{2} + \frac{1}{2}c\left[(1)^2 - 2 \cdot \frac{1}{6}\right] + \frac{1}{2}d\left[(1)^3 - 3 \cdot \frac{1}{6}\right]$$

$$= a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$$

26. (a) : Here, $\tan\alpha + \tan\beta = p$ ...(i)

$\tan\alpha \cdot \tan\beta = q$ ...(ii)

Hence, $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{p}{1 - q}$

Now, $\sin^2(\alpha + \beta) = \dfrac{1 - \cos[2(\alpha + \beta)]}{2}$

$$= \frac{1}{2}\left\{1 - \left(\frac{1 - \tan^2(\alpha + \beta)}{1 + \tan^2(\alpha + \beta)}\right)\right\} = \frac{1}{2}\left[1 - \frac{1 - \left(\frac{p}{1 - q}\right)^2}{1 + \left(\frac{p}{1 - q}\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(1 - q)^2 + p^2 - (1 - q)^2 + p^2}{(1 - q)^2 + p^2}\right] = \frac{p^2}{p^2 + (1 - q)^2}.$$

27. (a) : Let the roots are α, β.

Since, $\frac{\alpha}{\beta}=\frac{2}{3}\Rightarrow \alpha=\frac{2\beta}{3}$

Also, $\alpha+\beta=\frac{m}{12} \Rightarrow \frac{2\beta}{3}+\beta=\frac{m}{12}\Rightarrow \frac{5\beta}{3}=\frac{m}{12}$ ...(i)

and $\alpha\beta=\frac{5}{12}\Rightarrow \frac{2\beta}{3}\cdot\beta=\frac{5}{12}\Rightarrow\beta^2=\frac{5}{8}$

$\Rightarrow \beta=\sqrt{5/8}$

Put the value of β in (i), $\frac{5}{3}\cdot\sqrt{\frac{5}{8}}=\frac{m}{12} \Rightarrow m=5\sqrt{10}\cdot$

28. (d) : $(\alpha^2+\beta^2)^2=(\alpha+\beta)(\alpha^3+\beta^3)$

$$\left(\frac{b^2-2ac}{a^2}\right)^2=\left(\frac{-b}{a}\right)\left(\frac{-b^3+3abc}{a^3}\right)$$

$\Rightarrow 4a^2c^2=acb^2 \Rightarrow ac(b^2-4ac)=0$

As $a\neq 0\Rightarrow c\Delta=0$

29. (a,b) : let α, α^2 be the roots. Then, $\alpha+\alpha^2=30$ and $\alpha^3=p$

$\Rightarrow \alpha^2+\alpha-30=0 \Rightarrow (\alpha+6)(\alpha-5)=0$

$\Rightarrow \alpha=-6, 5$

$\therefore p=\alpha^3=(-6)^3=-216$ and $p=(5)^3=125$

$\Rightarrow p=125$ and -216.

30. (b,d) : Since roots of the equation $x^2-5x+16=0$ are α, β.

$\therefore \alpha+\beta=5$ and $\alpha\beta=16$

Also, $\alpha^2+\beta^2+\frac{\alpha\beta}{2}=-p$

$\Rightarrow (\alpha+\beta)^2-2\alpha\beta+\frac{\alpha\beta}{2}=-p \Rightarrow -p=25-32+8$

$\Rightarrow p=-1.$

Now, $(\alpha^2+\beta^2)\left(\frac{\alpha\beta}{2}\right)=q$

$\Rightarrow [(\alpha+\beta)^2-2\alpha\beta]\left[\frac{\alpha\beta}{2}\right]=q$

$\Rightarrow q=[25-32]\frac{16}{2}=-56$

So, $p=-1$, $q=-56$.

31. (a, b) : Let α and β are the roots of $Ax^2+Bx+C=0$

$\because |ax^2+bx+c|\leq|Ax^2+Bx+C|, \forall x\in R$

$\Rightarrow ax^2+bx+c=0$ also has α, β as roots.

$\Rightarrow |ax^2+bx+c|=|a|\,|x-\alpha|\,|x-\beta|\leq (Ax^2+Bx+C|$

$=|A|\,|x-\alpha|\,|x-\beta|$

$\Rightarrow |a|\leq|A|$

and $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$\Rightarrow \frac{b^2-4ac}{a^2}=\frac{B^2-4AC}{A^2}\Rightarrow |d|\leq|D|$

32. (c, d) : Let $P=3\alpha+2$

$\Rightarrow \alpha=\frac{P-2}{3}$

Since, $9\alpha^3-7\alpha+6=0$

$\Rightarrow \frac{9(P-2)^3}{27}-\frac{7}{3}(P-2)+6=0$

$\Rightarrow \frac{1}{3}(P^3-8-6P^2+12P)-\frac{7}{3}P+\frac{14}{3}+6=0$

$\Rightarrow P^3-6P^2+12P-8-7P+14+18=0$

$\Rightarrow P^3-6P^2+5P+24=0$

So, the equation $x^3-6x^2+5x+24=0$ has roots $3\alpha+2$, $3\beta+2$, $3\gamma+2$

$\Rightarrow A=-6$, $B=5$ and $C=24$

(33 – 34) :

$f(x)=ax^2+bx+c$, $a\neq0$, $a, b, c\in I$

$f(1)=0$

$\Rightarrow a+b+c=0$ ...(i)

Also, $50<f(7)<60$

$\Rightarrow 50<49a+7b+c<60$

$\Rightarrow 50<48a+6b<60$

$\Rightarrow \frac{50}{6}<8a+b<10$

$\Rightarrow 8a+b=9$ ...(ii)

Also, $70<f(8)<80$

$\Rightarrow 70<64a+8b+c<80$

$\Rightarrow 70<63a+7b<80$

$\Rightarrow 10<9a+b<\frac{80}{7}$

$\Rightarrow 9a+b=11$ ...(iii)

From (i), (ii) and (iii), $a=2$, $b=-7$, $c=5$

33. (c) : $f(x)=2x^2-7x+5$

$$=2\left(x^2-\frac{7}{2}x+\frac{5}{2}\right)$$

$$=2\left(\left(x-\frac{7}{4}\right)^2-\frac{9}{16}\right)$$

$\Rightarrow f(x)$ has least value $-\frac{9}{8}\cdot$

34. (b) : Since, $f(x)<0\Rightarrow 2x^2-7x+5$

$=(2x-5)(x-1)<0$

$\Rightarrow 1<x<5/2$

(35 – 36) :

Let us assume that $A\equiv(\alpha,0)$, $B\equiv(2\alpha,0)$, $C\equiv(0,2\alpha)$.

Now $y=x^2+bx+c$ passes through $C(0,2\alpha)$.

Thus, given equation of curve reduces to
$y = x^2 + bx + 2\alpha$.
Now, it also passes through A and B
$\therefore \quad 0 = \alpha^2 + b\alpha + 2\alpha$
$\Rightarrow \quad 0 = \alpha + b + 2$ ...(i)
and $0 = 4\alpha^2 + 2\alpha b + 2\alpha$
$\Rightarrow \quad 0 = 2\alpha + b + 1$ ...(ii)
On solving (i) and (ii) for α and b, we get
$\alpha = 1, b = -3$
Thus, given curve is $y = x^2 - 3x + 2$
Roots of $y = 0$ are $\{2, 1\}$.

35. (a) : Since $\alpha = 2, \beta = 1$
$\therefore \quad (\alpha + \beta) = 3$
$(\alpha - \beta) = 1$
Hence, equation whose roots are 3 and 1 is :
$x^2 - 4x + 3 = 0$

36. (d) : Minimum value of $y = x^2 - 4x + 3$ occurs at
$x = \frac{-b}{2a}$ or $x = \frac{-(-4)}{2} = 2$

37. A $\rightarrow$ (q, r) ; B $\rightarrow$ (q, s); C $\rightarrow$ (s)
(A) Discriminant $= 1 + 4a$
$1 + 4a$ must be a perfect square.
Let $1 + 4a = (2\lambda + 1)^2; \lambda \in N$
$\Rightarrow \quad a = \lambda(\lambda + 1)$ (as $1 + 4a$ is odd)
(B) $f(x) = ax^2 + 2bx + 4c - 16 = 0$
$f(-2) = 4a - 4b + 4c - 16$
$\Rightarrow \quad f(-2) = (4a + c - b - 4)$
$f(-2) > 0$ as $a + c > b + 4$
$\Rightarrow \quad f(0) > 0 \quad \Rightarrow \quad 4c - 16 > 0$
$\Rightarrow \quad c > 4$
(C) As both roots are negative so
$2b > 0$ and $9b - 14 > 0$ ...(i)
$\Rightarrow \quad D \geq 0 \Rightarrow 4b^2 - 4(9b - 14) \geq 0$
$\Rightarrow \quad b^2 - 9b + 14 \geq 0$
$\Rightarrow \quad (b - 2)(b - 7) \geq 0$
$\Rightarrow \quad b \in (-\infty, 2] \cup [7, \infty]$ ...(ii)
Intersection (i) and (ii) is $b \in \left(\frac{14}{9}, 2\right) \cup [7, \infty)$

38. (1) : Let α be the root of the given cubic equation where α can take values a, b or c.
Hence, $\alpha^3 + \alpha^2 - 333\alpha - 1002 = 0$
or $\quad \alpha^3 = 1002 + 333\alpha - \alpha^2$
$\therefore \quad \Sigma\alpha^3 = \Sigma 1002 + 333\Sigma\alpha - \Sigma\alpha^2$
$= 3006 + 333\Sigma\alpha - [(\Sigma\alpha)^2 - 2\Sigma\alpha_1\alpha_2]$
But $\Sigma\alpha = -1; \Sigma\alpha_1\alpha_2 = -333$
$\therefore \quad a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666]$
$= 3006 - 333 - 667 = 3006 - 1000 = 2006 = P$

39. (2) : $f'(x) = -2 - 3x^2 < 0 \Rightarrow f(x)$ is decreasing
$\therefore \quad f(f(x)) < f(-x) \Rightarrow f(x) > -x$
$\Rightarrow \quad 30 - x - x^3 > 0 \Rightarrow x^3 + x - 30 < 0$
$\Rightarrow \quad (x - 3)(x^2 + 3x + 10) < 0$
$\Rightarrow \quad x < 3$
$\therefore \quad$ Number of values $= 2$

40. (0) : $ax^2 + 2cx + b = 0$ ...(i)
$ax^2 + 2bx + c = 0$ ...(ii)
(i) and (ii) imply that
$\Rightarrow \quad \frac{x^2}{2(c^2 - b^2)} = \frac{x}{a(b-c)} = \frac{1}{2a(b-c)}$
$\Rightarrow \quad 4a(c^2 - b^2)(b - c) = a^2(b - c)^2$
$\Rightarrow \quad -4(b + c) = a \Rightarrow a + 4b + 4c = 0.$

Good posture can boost students' Math scores

A change of body posture can help boost a student's Math scores, especially if they are known to fear the subject, a study has found.

Researchers at San Francisco State University tested 125 college students to see how well they could perform simple Math — subtracting 7 from 843 sequentially for 15 seconds — while either slumped over or sitting up straight with shoulders back and relaxed.

Fifty-six per cent of the students reported finding it easier to perform the Math in the upright position. "For people who are anxious about Math, posture makes a giant difference," said Erik Peper, from San Francisco State University.

"The slumped-over position shuts them down and their brains do not work as well. They cannot think as clearly," said Peper.

Before the study began, students filled out a questionnaire asking them to rate their anxiety levels while performing Math; they also described any physical symptoms of stress they experienced during test taking.

Slumping over is a defensive posture that can trigger old negative memories in the body and brain, according to associate professor Richard Harvey. While the students without Math anxiety did not report as great a benefit from better posture, they did find that doing Math while slumped over was more difficult.

The findings can help people prepare for many different types of performance under stress, not just Math tests, researchers said.

Athletes and public speakers can also benefit from better posture prior to and during their performance. This could be helpful to students facing the challenge called "stereotype threat," said Lauren Mason, a SF State graduate.

Courtesy : The Times of India

CLASS XI | Series 5

CBSE DRILL

Synopsis and Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Permutations and Combinations | Binomial Theorem

Permutations and Combinations

FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication rule

If a work can be done in m different ways and another work can be done in n different ways, then number of ways of doing the two jobs in succession is $m \times n$.

Addition rule

If a work can be done in m different ways and another work can be done in n different ways, then either of the two works can be done in $(m + n)$ ways.

FACTORIAL NOTATION

Let n be a positive integer. Then product of first n natural numbers is defined as n factorial. It is denoted by $n!$. Thus, $n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$

Also, we defined $0 ! = 1$

PERMUTATIONS

The different arrangements of given number of objects that can be made by taking some or all at a time.

Formulae Used To find the number of permutations in different situation

Situations		Formula used
When r things are taken out from n different things at a time	If repetition is not allowed	${}^{n}P_{r}$ *i.e.*, $\frac{n!}{(n-r)!}$
	If repetition is allowed	n^r
When all the things are taken out from n different things at a time		${}^{n}P_{n}$ *i.e.*, $n!$
Out of n objects, p are of same kind and the rest are all different		$\frac{n!}{p!}$
Out of n objects, p_1 are of first kind, p_2 are of second kind, …, p_k are of k^{th} kind and rest are all different		$\frac{n!}{p_1! p_2! \ldots p_k!}$

COMBINATIONS

Selections which can be made by taking some or all number of objects, irrespective of their arrangement is defined as combination.

- Number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ and is defined by

$${}^nC_r = \frac{n!}{(n-r)!r!}, \text{ where } 0 \le r \le n.$$

Some Useful Results

- For $0 \le r \le n$, ${}^nC_r = {}^nC_{n-r}$
- For $1 \le r \le n$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nP_r = {}^nC_r \cdot r!$, $0 < r \le n$
- If ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x = n - y$

Binomial Theorem

Definition

The formula by which any power of a binomial expression can be expanded in the form of a series is known as binomial theorem. When n is a positive integer (natural number), then

$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + {}^nC_nb^n$

or $$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r}b^r$$

Here, ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, …, ${}^nC_n$ are known as binomial coefficients, which is denoted by $C_0, C_1, C_2, \ldots, C_n$.

Note : In the expansion of $(a+b)^n$

(i) The number of terms in the expansion is $n+1$ *i.e.* one more than the index n.

(ii) Sum of indices of a and b in each term of binomial expansion is n.

(iii) r^{th} term from the end is $(n-r+2)^{th}$ term from the beginning.

Some other expansions

- $(a-b)^n = {}^nC_0a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 - \ldots + (-1)^n {}^nC_nb^n$
- $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$
- $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \ldots + (-1)^n {}^nC_nx^n$

General and middle terms

General Term: In the expansion of $(a+b)^n$,

General term = T_{r+1} *i.e.*, $(r+1)^{th}$ term = ${}^nC_r\, a^{n-r} \cdot b^r$

Middle Term: In the expansion of $(a+b)^n$, the middle term is given by

- $\left(\frac{n}{2}+1\right)^{th}$ term, if n is even.
- $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term, if n is odd.

WORK IT OUT

VERY SHORT ANSWER TYPE

1. Prove that $20! = 2^{10}[1 \cdot 3 \cdot 5 \ldots \cdot 19] \cdot 10!$
2. If $(n+1)! = 6[(n-1)!]$, then find n.
3. Expand $(x^2 + 2a)^5$ by binomial theorem.
4. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 choices each?
5. Prove that, $P(n, n) = 2\,P(n, n-2)$

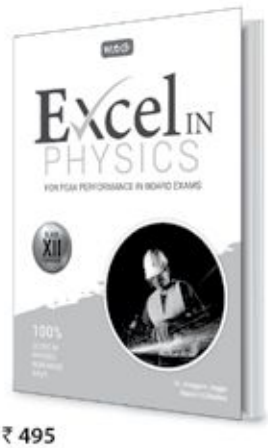
SHORT ANSWER TYPE

6. Find the coefficient of x^9 in $(1 + 3x + 3x^2 + x^3)^{15}$.
7. Find the 5^{th} term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.
8. For the post of 5 teachers, there are 23 applicants. 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?
9. How many two digit odd numbers can be formed from the digits 1, 3, 4, 5 and 8, if
   (i) repetition of digits is allowed
   (ii) repetition of digits is not allowed.
10. Find the value of r, if the coefficients of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal.

LONG ANSWER TYPE - I

11. Find the middle term and term independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$.
12. Find n, if
   (i) ${}^nP_4 = 18 \cdot {}^{n-1}P_2$ (ii) ${}^nP_5 : {}^{n-1}P_4 = 6 : 1$
13. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done, if atleast five women have to be included in a committee? In how many of these committees
   (i) the women are in majority and
   (ii) the men are in majority?

Concerned about your performance in Class XII Boards?



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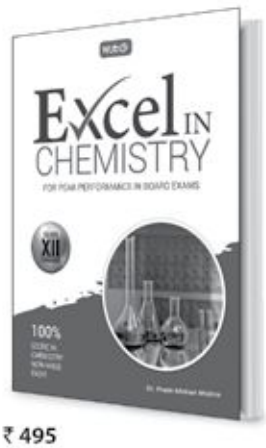

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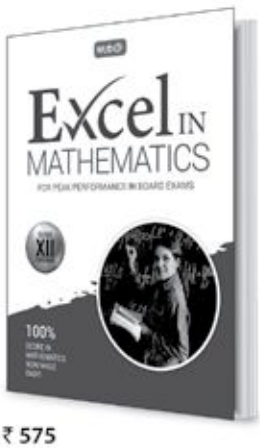

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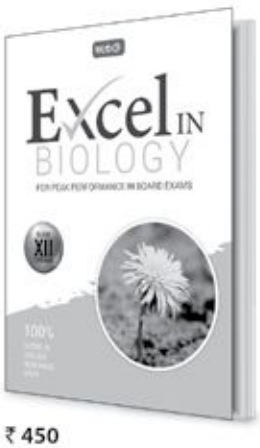

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14. How many different words can be formed of the letters of the word 'MALENKOV' so that
(i) no two vowels are together;
(ii) the relative position of the vowels and consonants remains unaltered?

15. If n be a positive integer, then prove that $6^{2n} - 35n - 1$ is divisible by 1225.

LONG ANSWER TYPE - II

16. Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

17. Find the greatest term in the expansion of $(7 - 5x)^{11}$, where $x = \frac{2}{3}$.

18. If three consecutive coefficients in the expansion of $(1 + x)^n$ be 165, 330 and 462, then find n and the position of the coefficient.

19. In how many ways can a cricket team of 11 players be selected out of 16 players
(i) if two particular players are always to be included?
(ii) if one particular player is to be excluded?
(iii) if two particular players are to be included and one particular player is to be rejected?

20. How many four-letter words can be formed using the letter of the word 'INEFFECTIVE'?

SOLUTIONS

1. $20! = 1 \cdot 2 \cdot 3 \ldots 18 \cdot 19 \cdot 20 = (2 \cdot 4 \cdot 6 \ldots \cdot 20) \cdot (1 \cdot 3 \cdot 5 \ldots \cdot 19)$
$= 2^{10}(1 \cdot 2 \cdot 3 \ldots \cdot 10)\,(1 \cdot 3 \cdot 5 \ldots \cdot 19)$
$= 2^{10} \times 10!\,(1 \cdot 3 \cdot 5 \ldots \cdot 19)$

2. $(n + 1)! = 6[(n - 1)!]$
$\Rightarrow (n + 1) \cdot n \cdot [(n - 1)!] = 6[(n - 1)!]$
$\Rightarrow n^2 + n = 6 \Rightarrow n^2 + n - 6 = 0$
$\Rightarrow (n - 2)(n + 3) = 0$
$\therefore$ Either $n - 2 = 0$ or $n + 3 = 0 \Rightarrow n = 2$ or $n = -3$
Since, n being natural number, so $n \neq -3$,
$\therefore \quad n = 2.$

3. $(x^2 + 2a)^5 = {}^5C_0\,(x^2)^5\,(2a)^0 + {}^5C_1\,(x^2)^4\,(2a)^1 + {}^5C_2\,(x^2)^3\,(2a)^2 + {}^5C_3\,(x^2)^2\,(2a)^3 + {}^5C_4\,(x^2)^1\,(2a)^4 + {}^5C_5\,(x^2)^0\,(2a)^5$
$= x^{10} + 5(x^8)\,(2a) + 10(x^6)\,(4a^2) + 10(x^4)\,(8a^3) + 5(x^2)\,(16a^4) + 32a^5$
$= x^{10} + 10x^8a + 40x^6a^2 + 80x^4a^3 + 80x^2a^4 + 32a^5$

4. Each of the first three questions can be answered in 4 ways and each of the last three questions can be answered in 5 ways.
By the fundamental principle of counting, sequences of answers are :
$4 \times 4 \times 4 \times 5 \times 5 \times 5 = 64 \times 125 = 8000$

5. $P(n, n) = n!$

$$2P(n, n-2) = 2 \cdot \frac{n!}{[n-(n-2)]!} = 2 \cdot \frac{n!}{2!} = n!$$

Hence $P(n, n) = 2P(n, n - 2)$.

6. $(1 + 3x + 3x^2 + x^3)^{15} = [(1 + x)^3]^{15} = (1 + x)^{45}$
$\therefore$ Coefficient of x^9 in $(1 + 3x + 3x^2 + x^3)^{15}$
= Coefficient of x^9 in $(1 + x)^{45}$
$= {}^{45}C_9 = \frac{45!}{9!36!}$

7. We know that r^{th} term from the end = $(n - r + 2)^{\text{th}}$ term from the beginning.
We have, $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$
$\therefore$ 5^{th} term from the end = $(9 - 5 + 2)^{\text{th}}$ term from the beginning = 6^{th} term from the beginning = T_6

$$\therefore \quad T_6 = (-1)^5 \cdot {}^9C_5 \left(\frac{x^3}{2}\right)^{9-5} \left(\frac{2}{x^2}\right)^5$$

$$= -{}^9C_4 \left(\frac{x^3}{2}\right)^4 \left(\frac{2}{x^2}\right)^5 \qquad [\because {}^9C_5 = {}^9C_4]$$

$$= -\left(\frac{9\times8\times7\times6}{4\times3\times2\times1} \times \frac{x^{12}}{16} \times \frac{32}{x^{10}}\right) = -252\,x^2.$$

8. Clearly, there are 7 SC candidates and 16 non-SC candidates.
So, we have to make a selection of (2 out of 7) and (3 out of 16).
$\therefore$ The number of ways of making the selection

$$= {}^7C_2 \times {}^{16}C_3 = \left(\frac{7\times6}{2} \times \frac{16\times15\times14}{3\times2\times1}\right) = 11760.$$

9. The number is odd if 1 or 3 or 5 appears in the unit's place. Therefore, there are three ways of filling the unit's place.
(i) Since repetition of digits is allowed, so the ten's place can be filled by any of the five digits 1, 3, 4, 5 and 8. Hence, number of two digit odd numbers = $3 \times 5 = 15$
(ii) Since repetition of digits is not allowed, so the ten's place can be filled by any of the remaining four digits. Hence, number of two digit odd numbers = $3 \times 4 = 12$

10. We have
$T_{2r+4} = T_{(2r+3)+1} = {}^{18}C_{2r+3}\,x^{2r+3}$;
and $T_{r-2} = T_{(r-3)+1} = {}^{18}C_{r-3}\,x^{r-3}$.
Since, coefficient of $T_{(2r+4)}$ = Coefficient of $T_{(r-2)}$
$\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow$ $2r + 3 = r - 3$ or $2r + 3 + r - 3 = 18$

$\Rightarrow$ $r = -6$ or $r = 6$.

But, the negative value of r is not possible, so $r = 6$.

11. General term in $\left(2x^2 - \frac{1}{x}\right)^7 = {}^7C_r\,(2x^2)^{7-r}\left(-\frac{1}{x}\right)^r$

Here $n = 7$, which is an odd number.

Therefore, $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+3}{2}\right)^{th}$ *i.e.*, 4^{th} and 5^{th} terms will be the two middle terms.

Now, 4^{th} term in $\left(2x^2 - \frac{1}{x}\right)^7 = {}^7C_3\,(2x^2)^{7-3}\left(-\frac{1}{x}\right)^3$

$= {}^7C_3\,(2)^4x^8.\ \frac{(-1)^3}{x^3} = \frac{7!}{3!4!} \times 16x^5(-1) = -560x^5$

and 5^{th} term $= {}^7C_4\,(2x^2)^{7-4}\left(-\frac{1}{x}\right)^4$

$= 35.(2)^3\,x^6\,.\ \frac{(-1)^4}{x^4} = 280x^2$

Hence, the two middle terms are $-560x^5$ and $280x^2$.

Now, general term $= {}^7C_r\,(2)^{7-r}\,(x)^{14-2r}\,(x)^{-r}\,(-1)^r$

For term to be independent of x,

$14 - 3r = 0 \Rightarrow r = \frac{14}{3}$, which is not possible.

Hence, there is no term independent of x.

12. (i) ${}^nP_4 = 18 \cdot {}^{n-1}P_2$

$\Rightarrow \frac{n!}{(n-4)!} = 18 \cdot \frac{(n-1)!}{(n-1-2)!}$

$\Rightarrow \frac{n!}{(n-4)!} = 18 \cdot \frac{(n-1)!}{(n-3)!}$

$\Rightarrow \frac{n(n-1)!}{(n-4)!} = 18 \cdot \frac{(n-1)!}{(n-3)(n-4)!}$

$\therefore\ n = \frac{18}{n-3} \Rightarrow n(n-3) = 18$

$\Rightarrow n^2 - 3n - 18 = 0$

$\Rightarrow (n-6)(n+3) = 0 \Rightarrow n = 6, -3$

But n cannot be negative.

$\therefore\ n = 6$

(ii) $\frac{{}^nP_5}{{}^{n-1}P_4} = \frac{6}{1}$

$\Rightarrow \frac{n!}{(n-5)!} \times \frac{(n-1-4)!}{(n-1)!} = \frac{6}{1}$

$\Rightarrow \frac{n(n-1)! \times (n-5)!}{(n-5)! \times (n-1)!} = 6$

$\Rightarrow n = 6$

13. The committee can be selected in the following ways:

Men	7	6	5	4	3
Women	5	6	7	8	9

Number of ways in which 7 men and 5 women can be selected

$= {}^8C_7 \times {}^9C_5 = {}^8C_1 \times {}^9C_4 = 8 \times \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 1008$

Number of ways in which 6 men and 6 women can be selected

$= {}^8C_6 \times {}^9C_6 = {}^8C_2 \times {}^9C_3 = \frac{8 \cdot 7}{1 \cdot 2} \times \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 2352$

Number of ways in which 5 men and 7 women can be selected

$= {}^8C_5 \times {}^9C_7 = {}^8C_3 \times {}^9C_2 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \times \frac{9 \cdot 8}{1 \cdot 2} = 2016$

Number of ways in which 4 men and 8 women can be selected

$= {}^8C_4 \times {}^9C_8 = {}^8C_4 \times {}^9C_1 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \times 9 = 630$

Number of ways in which 3 men and 9 women can be selected

$= {}^8C_3 \times {}^9C_9 = {}^8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$

Therefore, total number of ways = 1008 + 2352 + 2016 + 630 + 56 = 6062

(i) Number of committees in which women are in majority = 2016 + 630 + 56 = 2702

(ii) Number of committees in which men are in majority = 1008

14. (i) Number of letters in the word 'MALENKOV' = 8

Number of vowels in the word 'MALENKOV' = 3

(i) Since, no two vowels are together, so first we arrange the consonants

$\times\ C \times C \times C \times C \times C \times$

The five consonants can be arranged in ${}^5P_5$ ways and the vowels A, E, O can be arranged in the 6 '×' marked places in ${}^6P_3$ ways.

$\therefore$ Total number of different words formed $= {}^5P_5 \times {}^6P_3$

$= 5! \times \frac{6!}{3!} = 120 \times 6 \times 5 \times 4 = 14400$

(ii) The relative positions of the vowels and consonants remain unaltered which means a vowel can take the place of the other vowel and a consonants that of the other consonant.

Now, 3 vowels can be arranged in ${}^3P_3$ ways, and 5 consonants can be arranged $= {}^5P_5$ ways.

$\therefore$ Required number of words $= {}^3P_3 \times {}^5P_5$

$= 3! \times 5! = 6 \times 120 = 720$.

15. Given expression $= 6^{2n} - 35n - 1$
$= (6^2)^n - 35n - 1 = (36)^n - 35n - 1$
Now $(36)^n - 35n - 1 = (1+35)^n - 35n - 1$
$= 1 + 35n + {}^nC_2 \cdot 35^2 + {}^nC_3 \cdot 35^3 + \ldots + 35^n - 35n - 1$
$[\because {}^nC_1 = n, {}^nC_n = 1 \text{ and } {}^nC_0 = 1]$
$= 35^2 [{}^nC_2 + {}^nC_3 \cdot 35 + \ldots + 35^{n-2}] + [1 + 35n - 35n - 1]$
$= 1225 [{}^nC_2 + {}^nC_3 \cdot 35 + \ldots + 35^{n-2}]$
= 1225 × a positive integer, if $n \geq 2$
If $n = 1$, then $6^{2n} - 35n - 1 = 36 - 35 - 1 = 0$, which is divisible by 1225.
Hence for all positive integral values of n, $6^{2n} - 35n - 1$, is divisible by 1225.

16. The expansion of $(1 + x)^{2n}$ contains $(2n + 1)$ terms and therefore, the $(n + 1)^{th}$ term is the middle term.
$\therefore$ The middle term in the expansion of $(1 + x)^{2n}$ is given by

$$T_{n+1} = {}^{2n}C_n x^n = \frac{(2n)!}{(n!)\cdot(n!)} x^n = \frac{(2n)!}{(n!)^2} x^n.$$

Thus, the coefficient of the middle term of $(1 + x)^{2n}$ is $\frac{(2n)!}{(n!)^2}$.
Again, the expansion of $(1 + x)^{2n-1}$ contains $2n$ terms.
So, the middle terms are the n^{th} and the $(n + 1)^{th}$ terms.
Now, $T_n = T_{(n-1)+1} = {}^{2n-1}C_{n-1}\, x^{n-1}$
and $T_{n+1} = {}^{2n-1}C_n x^n$.
$\therefore$ The sum of the coefficients of the middle terms of $(1 + x)^{2n-1}$

$$= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$$

$$= \frac{(2n-1)!}{(n-1)!\times(n!)} + \frac{(2n-1)!}{(n!)\times(n-1)!}$$

$$= \frac{2\times(2n-1)!\times n}{(n!)\times n\times(n-1)!} \quad \text{[Multiplying num. and denom. by } n\text{]}$$

$$= \frac{(2n)\cdot(2n-1)!}{(n!)\cdot(n!)} = \frac{(2n)!}{(n!)^2}$$

= The coefficient of the middle term of $(1 + x)^{2n}$.

17. Since we have to find numerically greatest term.
$\therefore$ Greatest term in $(7 - 5x)^{11}$ = Greatest term in $(7 + 5x)^{11}$
Let r^{th} term be the greatest term in the expansion of $(7 + 5x)^{11}$
Now, $T_r = {}^{11}C_{r-1}\,(7)^{11-r+1}\,(5x)^{r-1}$
$= {}^{11}C_{r-1}\, 7^{12-r}\,(5x)^{r-1}$ …(1)
and $(r + 1)^{th}$ term, $T_{r+1} = {}^{11}C_r\,(7)^{11-r}\,(5x)^r$ …(2)
$\therefore$

$$\frac{T_r}{T_{r+1}} = \frac{{}^{11}C_{r-1}}{{}^{11}C_r} \cdot \frac{7^{12-r}(5x)^{r-1}}{7^{11-r}(5x)^r}$$

$$= \frac{11!}{(r-1)!(12-r)!} \times \frac{r!(11-r)!}{11!}\left(\frac{7}{5x}\right)$$

or $\frac{T_r}{T_{r+1}} = \frac{r}{12-r}\left(\frac{7}{5.2/3}\right) = \frac{21r}{(12-r)10} \quad \left[\because x = \frac{2}{3}\right]$ …(3)

Since r^{th} term is the greatest term
$\therefore \quad T_r \geq T_{r+1}$ or $\frac{T_r}{T_{r+1}} \geq 1 \Rightarrow \frac{21r}{(12-r)10} \geq 1$
or $21r \geq 120 - 10r$
or $31r \geq 120$ or $r \geq \frac{120}{31}$ or $r \geq 3\frac{27}{31}$ …(A)
Putting $(r - 1)$ in place of r in (3),

$$\frac{T_{r-1}}{T_r} = \frac{21(r-1)}{\{12-(r-1)\}\cdot 10} = \frac{21r-21}{130-10r}$$

$\therefore \quad \frac{T_r}{T_{r-1}} = \frac{130-10r}{21r-21}$
$\because \quad \frac{T_r}{T_{r-1}} \geq 1 \Rightarrow \frac{130-10r}{21r-21} \geq 1$ or $130 - 10r \geq 21r - 21$
$\therefore \quad r \leq \frac{151}{31}$ or $r \leq 4\frac{27}{31}$ …(B)
From (A) and (B), we have $3\frac{27}{31} \leq r \leq 4\frac{27}{31} \Rightarrow r = 4$
Hence 4^{th} term is the greatest term.
From (1), greatest term $T_4 = {}^{11}C_3 . 7^8 \left(5.\frac{2}{3}\right)^3$

$$= \frac{11\cdot 10\cdot 9}{6} \times 7^8 5^3 \times \frac{8}{27} = \frac{440}{9} . 7^8 5^3$$

18. Let the three consecutive coefficients in the expansion of $(1 + x)^n$ be r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ which are given to be 165, 330 and 462 respectively.
$\therefore \quad {}^nC_{r-1} = 165$ or $\frac{n!}{(r-1)!(n-r+1)!} = 165$ …(1)

$${}^nC_r = 330 \text{ or } \frac{n!}{r!(n-r)!} = 330 \quad \ldots(2)$$

and ${}^nC_{r+1} = 462$ or $\frac{n!}{(r+1)!(n-r-1)!} = 462$ …(3)

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Dividing (1) by (2), we get

$$\frac{n!}{(r-1)!(n-r+1)!}\times\frac{r!(n-r)!}{n!}=\frac{165}{330}$$

or $\frac{r}{n-r+1}=\frac{1}{2}$ or $2r = n - r + 1$ or $3r = n + 1$ …(4)

Dividing (2) by (3), we get

$$\frac{n!}{r!(n-r)!}\times\frac{(r+1)!(n-r-1)!}{n!}=\frac{330}{462}$$

or $\frac{r+1}{n-r}=\frac{5}{7}$ or $7r + 7 = 5n - 5r$ or $12r = 5n - 7$ …(5)

On solving (4) and (5), we get $n = 11$ and $r = 4$ …(6)

Hence, $n = 11$ and the positions of the consecutive coefficients are 4^{th}, 5^{th} and 6^{th} respectively.

19. 11 players can be selected out of 16 players in ${}^{16}C_{11}$ ways $= \frac{16!}{11!5!} = 4368$ ways.

(i) When two particular players are always to be included, then 9 more players are to be selected out of remaining 14 players, which can be done in

${}^{14}C_9$ ways $= \frac{14!}{9!5!} = 2002$ ways

(ii) If one particular player is to be excluded, then selection is to be made of 11 players out of 15 players and this can be done in ${}^{15}C_{11}$ ways

$= \frac{15!}{11!4!} = 1365$ ways

(iii) If two particular players are to be included and one particular player is to be rejected, then we have to select 9 more out of 13 in ${}^{13}C_9$ ways

$= \frac{13!}{9!4!} = 715$ ways

20. There are 11 letters in the word 'INEFFECTIVE'. respectively EEE, FF, II, C, T, N, V.

The four-letter words may consist of:

(i) 3 alike letters and 1 distinct letter

(ii) 2 alike letters of one kind and 2 alike letters of the second kind

(iii) 2 alike letters and 2 distinct letters

(iv) all different letters

Now we shall discuss these four cases one by one:

(i) **3 alike letters and 1 distinct letter :** There is one set of three alike letters viz. EEE. So, three alike letters can be selected in one way. Out of the 6 different letters one letter can be selected in ${}^6C_1$ ways. These 4 letters can be arranged in $\frac{4!}{3!1!}$ ways.

Hence, the total number of words consisting of three alike and one distinct letters

$= {}^6C_1 \times \frac{4!}{3!1!} = {}^6C_1 \times 4 = 24.$

(ii) **2 alike letters of one kind and 2 alike letters of second kind :** There are three sets of two alike letters respectively EE, FF, II. Out of these three sets two can be selected in ${}^3C_2$ ways. Now, 4 letters in each group can be arranged in $\frac{4!}{2!2!}$ ways.

Hence, the total number of words consisting of two alike letters of one type and 2 alike letters of second type

$= {}^3C_2 \times \frac{4!}{2!2!} = 18$

(iii) **2 alike and 2 different letters:** Out of 3 sets of two alike letters one set can be chosen in ${}^3C_1$ ways. Now, from the remaining 6 distinct letters, 2 letters can be chosen in ${}^6C_2$ ways. Thus, 2 alike letters and 2 distinct letters can be selected in $({}^3C_1 \times {}^6C_2)$ ways. Now, letters of each group can be arranged among themselves in $\frac{4!}{2!}$ ways.

Hence, the total number of words consisting of two alike letters and 2 distinct letters

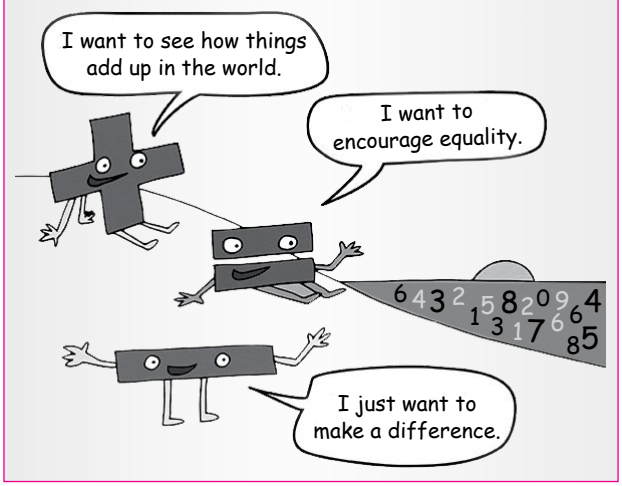
$= {}^3C_1 \times {}^6C_2 \times \frac{4!}{2!} = 540.$

(iv) **All different letters:** There are 7 distinct letters E, F, I, T, N, V, C out of which 4 can be selected in ${}^7C_4$ ways. The letters in each of ${}^7C_4$ groups can be arranged in 4! ways. So, the total number of 4 letter words in which all letters are distinct $= {}^7C_4 \times 4! = 840$.

Hence the total number of four letter words

$= 24 + 18 + 540 + 840 = 1422.$

COMIC CAPSULE



MPP-5 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.
Self check table given at the end will help you to check your readiness.

Binomial Theorem

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If each of the coefficient in the expansion of $(a + cx)^{-2}$ is positive, then quadratic equation $ax^2 + bx + c = 0$ has
(a) real and positive roots.
(b) real and negative roots.
(c) imaginary roots.
(d) real roots with opposite signs.

2. If $f(n) = \sum_{s=1}^{n}\sum_{r=s}^{n} {}^nC_r\ {}^rC_s$, then $f(3) =$
(a) 19 (b) 18
(c) 21 (d) none of these

3. If in the expansion of $\left(2^{\frac{1}{3}} + \frac{1}{3^{1/3}}\right)^n$, the ratio of the 7^{th} term from the beginning to the 7^{th} term from the end is 1 : 6, then $n =$
(a) 7 (b) 8 (c) 9 (d) 10

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is
(a) 120 (b) 210 (c) 310 (d) 4

5. The remainder when 17^{20} is divided by 81 is equal to
(a) 46 (b) 35
(c) 47 (d) none of these

6. The coefficient of x^2y in the expansion of $(1 + x + 2y)^5$ is
(a) 20 (b) 60
(c) 35 (d) 40

One or More Than One Option(s) Correct Type

7. Which of the following is true for the expansion of $(2 - 2x + x^2)^9$?
(a) Number of distinct terms is 10.
(b) Coefficient of x^4 is 97.
(c) Sum of coefficients is 1.
(d) Number of distinct terms is 55.

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is
(a) $2^{21} - 2^{10}$ (b) $2^{20} - 2^9$
(c) $2^{20} - 2^{10}$ (d) $2^{21} - 2^{11}$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then
(a) $a_1 = 20$ (b) $a_2 = 210$
(c) $a_4 = 8085$ (d) $a_{20} = 2^2 \cdot 3^7 \cdot 7$

10. Which of the following statements is/are true?
(a) $5^{2n} + 1$ is divisible by 13 if n is even.
(b) $5^{2n} - 1$ is divisible by 13 if n is odd.
(c) $5^{2n} - 1$ is divisible by 13 if n is even.
(d) $5^{2n} + 1$ is divisible by 13 if n is odd.

UPES launches six new undergraduate courses

University of Petroleum and Energy Studies (UPES) has launched six new UG courses designed to help students develop contemporary thinking and provide multi-dimensional problem-solving skills in each domain. Six new courses are - B.Com. (Hons), B.Com. (Hons) in Banking Management and Insurance, and in Taxation, B.Sc. in Chemistry, Mathematics and Physics. Candidates from Science, Commerce and Art stream with 50% in Class XII can apply for B.Com. (Hons) and other B.Com. courses. For B.Sc. (Hons) candidate should be from science stream with PCM with 50% in Class XII. Interested applicants can apply through www.upes.ac.in

◈◈

11. If the coefficients of x^{-2} and x^{-4} in the expansion of $\left(x^{\frac{1}{3}}+\frac{1}{2x^{\frac{1}{3}}}\right)^{18}$, $(x>0)$, are m and n respectively, then $\frac{m}{n}$ is equal to

(a) 27 (b) 182

(c) $\frac{5}{4}$ (d) $\frac{4}{5}$

12. If $S=C_0^2+\frac{C_1^2}{C_0}+2\cdot\frac{C_2^2}{C_1}+3\cdot\frac{C_3^2}{C_2}+\ldots+n\cdot\frac{C_n^2}{C_{n-1}}$ where $C_r={}^nC_r$, then

(a) $(n+2)$ divides $S+n$

(b) 2^{n-1} divides $S+n$

(c) S is even if n is even

(d) S is even for all n

13. If the coefficient of x^8 in the expansion of $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}\right)^2$ is $\frac{1}{M}$, then divisor(s) of M is/are

(a) 2 (b) 3 (c) 5 (d) 7

Comprehension Type

If n is a positive integer, then

$(x_1+x_2+x_3+\ldots+x_m)^n$

$=\sum\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_m!}\cdot x_1^{n_1}\cdot x_2^{n_2}\cdot x_3^{n_3}\ldots\ldots x_m^{n_m}$

where $n_1, n_2, n_3, \ldots\ldots, n_m$ are all non-negative integers subject to the condition $n_1+n_2+n_3+\ldots.+n_m=n$.

14. The coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ is

(a) 40 (b) 60 (c) 80 (d) 100

15. If coefficient of x^{20} in $(1-x+x^2)^{20}$ and $(1+x-x^2)^{20}$ are respectively a and b, then

(a) $a=b$ (b) $a>b$

(c) $a<b$ (d) $a+b=0$

Matrix Match Type

16. Match the following:

Column-I		Column-II	
P.	$\binom{10}{10}+\binom{11}{10}+\binom{12}{10}+\ldots+\binom{20}{10}$	1.	$\binom{19}{9}$
Q.	$11\cdot\binom{10}{10}+10\cdot\binom{11}{10}+9\cdot\binom{12}{10}+\ldots+1\cdot\binom{20}{10}$	2.	$\binom{20}{9}$
R.	$\binom{10}{0}\binom{10}{1}+\binom{10}{1}\binom{10}{2}+\binom{10}{2}\binom{10}{3}+\ldots+\binom{10}{9}\binom{10}{10}$	3.	$\binom{21}{10}$
S.	The number of non-negative solutions of $x_1+x_2+x_3+\ldots+x_{10}=10$ is	4.	$\binom{22}{10}$

	P	Q	R	S
(a)	1	3	4	2
(b)	3	2	4	1
(c)	3	4	2	1
(d)	1	3	2	4

Integer Answer Type

17. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n equals

18. If the sum of the two middle coefficients of $(1+x)^9$ is xyz then $x+y+z$ equals

19. The sum of the last 4 digits of the natural number 3^{100} is

20. The value of remainder when $(27)^{999}$ is divided by 7 is

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SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Functions

Class XII

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

DEFINITION

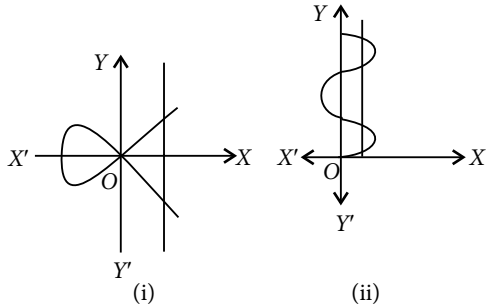
If A and B are two non-empty sets, then a rule f which associates to each $x \in A$, a unique number $y \in B$, is called a function from A to B and we write, $f : A \to B$. We say that 'y' is the image of 'x' under f (or x is the pre-image of y).

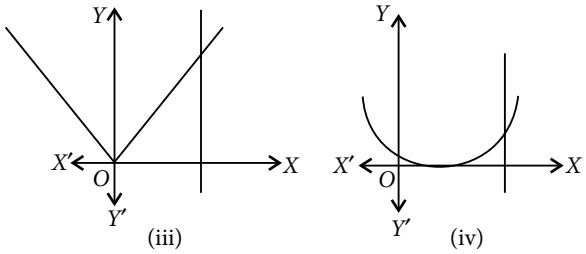
Note : 1. A mapping $f : X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

2. Functions can not be multi-valued *i.e.*, every element in set X should have one and only one image in Y.

Testing for a Function by Vertical Line Test

If it is possible to draw a vertical line which cuts the graph of given curve at more than one point, then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.





NUMBER OF FUNCTIONS

Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

If a function $f : A \to B$ is defined from a set A to set B, then set A is called the domain of function f and set B is called the co-domain of function f. The set of all f-images of the elements of A is called the range of function f.

Methods for finding Domain and Range of Function

- **Domain**

(i) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0.

(ii) Denominator $\neq 0$.

(iii) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$.

(iv) The domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

(v) Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

- **Range**

(i) If domain $\in$ finite number of points
$\Rightarrow$ Range $\in$ set of corresponding values of $f(x)$.

(ii) If domain $\in$ R or R – [some finite points], then express x in terms of y and find the values of y for which x exists.

(iii) If domain $\in$ a finite interval, then find the least and greatest value for range using monotonicity.

ALGEBRA OF FUNCTIONS

Let $f(x)$ and $g(x)$ be two real and single-valued functions with domains X_f, X_g and ranges Y_f and Y_g respectively. Then, the following operations are defined.

(i) Scalar multiplication of a function : $(cf)(x) = cf(x)$, where c is a scalar.

(ii) Addition/subtraction of functions : $(f \pm g)(x) = f(x) \pm g(x)$.

(iii) Multiplication of functions : $(fg)(x) = (gf)(x) = f(x)g(x)$.

(iv) Division of functions : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0.$

EQUAL FUNCTIONS

Two functions f and g are said to be equal functions, if and only if

(i) Domain of f = Domain of g

(ii) Co-domain of f = Co-domain of g

(iii) $f(x) = g(x)\ \forall\ x \in$ their common domain

KINDS OF FUNCTION

(i) One-one function (injectivity)

A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B.

Method to check the injectivity of a function

Step I : Take two arbitrary elements x, y (say) in the domain of f.

Step II : Put $f(x) = f(y)$

Step III : If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection), otherwise it is not.

Note : 1. If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

2. If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point, then function is one-one.

3. Any function, which is entirely increasing or decreasing in the whole of a domain is one-one.

- **Number of one-one functions (injections) :** If A and B are finite sets having m and n elements respectively, then number of one-one functions from A to $B = \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(ii) Many-one function

A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$

Note : 1. If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same, then function is many-one.

2. If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point, then function is many-one.

3. If the domain of the function is in one quadrant, then the trigonometric functions are always one-one.

4. If trigonometric function changes its sign in two consecutive quadrants, then it is one-one but if it does not change the sign, then it is many-one.

5. In three consecutive quadrants, trigonometric functions are always many-one.

6. Any continuous function $f(x)$, which has atleast one local maximum or local minimum is many-one.

(iii) Onto function (Surjectivity)

A function $f : A \to B$ is onto if each element of B has its pre-image in A.

In other words,

range of f = co-domain of f.

- **Number of onto functions (surjections) :** If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r r^m$.

(iv) Into function

A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A. In other words, $f : A \to B$ is an into function if it is not an onto function.

Method to find onto or into function

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say). Now, if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto.

Note : 1. An into function can be made onto by redefining the co-domain as the range of the original function.

2. Any polynomial function $f : R \to R$ is onto if degree of f is odd and into if degree of f is even.

(v) One-one Onto function (Bijection)

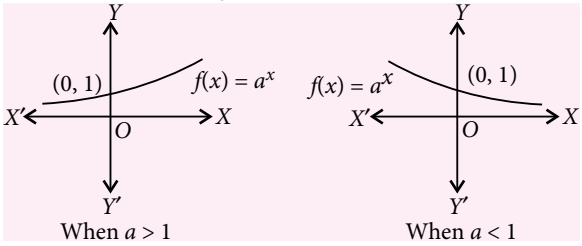
A function $f : A \to B$ is a bijection if it is one-one as well as onto. In other words, a function $f : A \to B$ is a bijection, if for $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$ and for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

- **Number of one-one onto functions (bijections) :** If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then number of bijections from A to B is $n!$

SOME SPECIAL FUNCTIONS

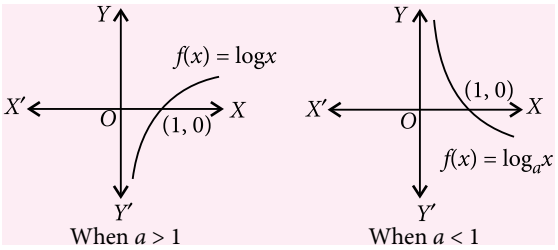
(i) Exponential function

Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.



(ii) Logarithmic function

Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.

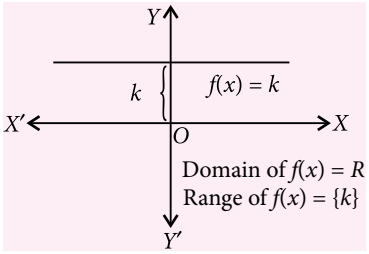


(iii) Explicit and implicit functions

A function is said to be explicit, if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit *e.g.* $y = \sin^{-1}x + \log x$ is an explicit function while $x^2 + y^2 = xy$ and $x^3y^2 = (a - x)^2(b - y)^2$ are implicit functions.

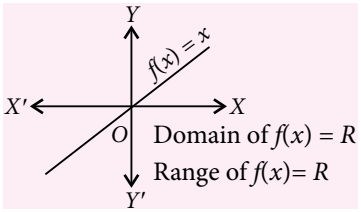
(iv) Constant function

Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.



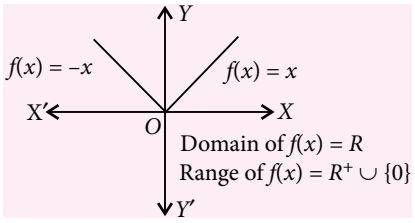
(v) Identity function

The function defined by $f(x) = x$ for all $x \in R$ is called the identity function on R.



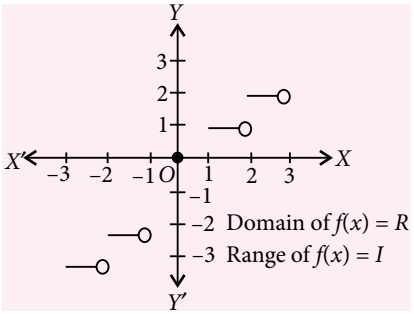
(vi) Modulus function

The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function.



(vii) Greatest integer function

The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function. Here $[x]$ denotes the greatest integer less than or equal to x for *e.g.* $[1{\cdot}1] = 1$, $[2{\cdot}2] = 2$, $[-0{\cdot}9] = -1$, $[-2{\cdot}1] = -3$ etc.

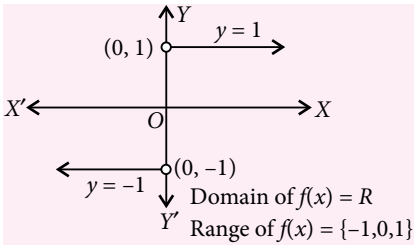


(viii) Signum function

The function defined by

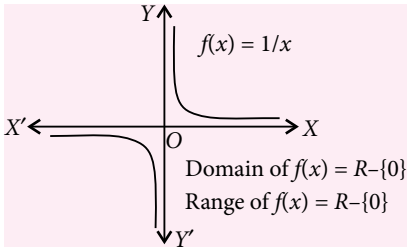
$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function.



(ix) Reciprocal function

The function that associates each non-zero real number x to the reciprocal $\frac{1}{x}$ is called the reciprocal function.



EVEN AND ODD FUNCTION

(i) Even function

If we put $(-x)$ in place of x in the given function and if $f(-x) = f(x)$, $\forall\ x \in$ domain, then $f(x)$ is called even function.

e.g. $f(x) = e^x + e^{-x}$, $f(x) = x^2$, $f(x) = x\sin x$, $f(x) = \cos x$ etc. are all even functions.

(ii) Odd function

If we put $(-x)$ in place of x in the given function and if $f(-x) = -f(x)$ $\forall\ x \in$ domain, then $f(x)$ is called odd function.

e.g., $f(x) = e^x - e^{-x}$, $f(x) = \sin x$, $f(x) = x^3$, $f(x) = x\cos x$, $f(x) = x^2 \sin x$ etc. are all odd functions.

Properties of Even and Odd Functions

- The graph of an even function is always symmetric with respect to y-axis.
- The graph of an odd function is always symmetric with respect to origin.
- The sum and difference of two even functions is an even function.
- The product of two even functions is an even function.
- The sum and difference of two odd functions is an odd function.
- The product of two odd functions is an even function.
- The product of an even and an odd function is an odd function
- It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function.
  e.g. $f(x) = x^2 + x^3$ and $f(x) = e^x$.
- The sum of an even and an odd function is neither even nor odd function.

PERIODIC FUNCTION

A function is said to be periodic function, if its each value is repeated after a definite interval. So, a function $f(x)$ will be periodic if there exists a positive real number T such that $f(x + T) = f(x)$, $\forall\ x \in$ domain. Here, the least positive value of T is called the period of the function.
Clearly $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = ...$
e.g. $\sin x$, $\cos x$, $\tan x$ are periodic functions with period 2π, 2π and π respectively.

Some standard results on periodic functions

	Functions	Periods
1.	$\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$	$\begin{cases}\pi, \text{ if } n \text{ is even}\\ 2\pi, \text{ if } n \text{ is odd or fraction}\end{cases}$
2.	$\tan^n x$, $\cot^n x$	π, n is even or odd
3.	$\lvert\sin x\rvert$, $\lvert\cos x\rvert$, $\lvert\tan x\rvert$, $\lvert\cot x\rvert$, $\lvert\sec x\rvert$, $\lvert\text{cosec}\, x\rvert$	π
4.	$x - [x]$	1

COMPOSITE FUNCTION

If $f : A \to B$ and $g : B \to C$ are two functions, then the composite function of f and g, given by $gof : A \to C$ will be defined as $gof(x) = g[f(x)]$ $\forall\ x \in A$.

Properties of Composition of Function

- f is even, g is even $\Rightarrow$ fog is even function.
- f is odd, g is odd $\Rightarrow$ fog is odd function.
- f is even, g is odd $\Rightarrow$ fog is even function.
- f is odd, g is even $\Rightarrow$ fog is even function.
- Composition of functions is not commutative, *i.e.*, $fog \neq gof$.
- Composition of functions is associative *i.e.*, $(fog)oh = fo(goh)$.
- $gof(x)$ is simply the g-image of $f(x)$, where $f(x)$ is f-image of elements $x \in A$.
- Function gof will exist only when range of f is the subset of domain of g.
- fog and gof may not be always defined. fog does not exist if range of g is not a subset of domain of f.
- If both f and g are one-one, then fog and gof are also one-one.
- If both f and g are onto, then gof is onto.

INVERSE OF A FUNCTION

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each

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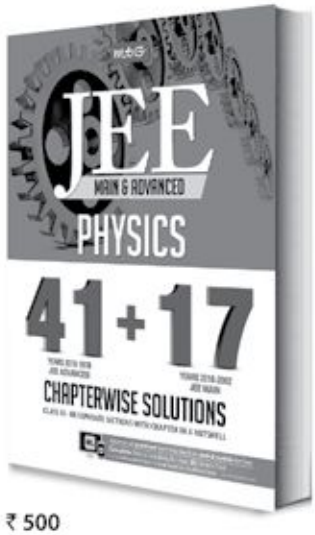
Mad about rehearsing?



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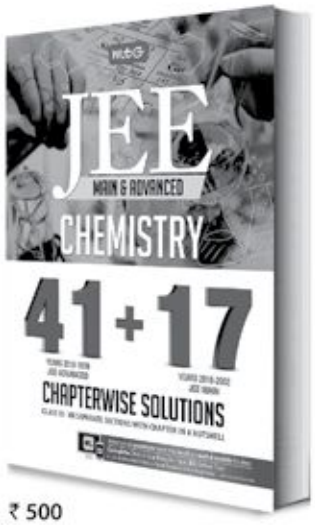

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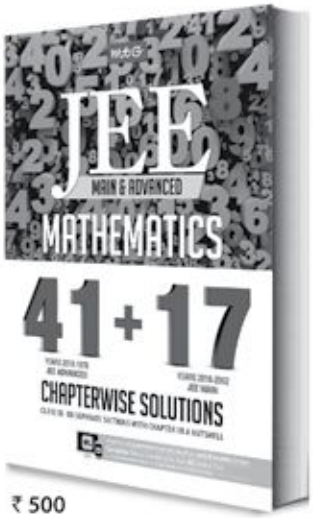

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element $b \in B$ with element $a \in A$ such that $f(a) = b$ is called the inverse of the function $f: A \to B$ and denoted by $f^{-1}: B \to A$ as $f^{-1}(b) = a \Rightarrow f(a) = b$

Important Points for Inverse Function

- For the existence of inverse function, it should be one-one and onto.
- Inverse of a bijection is also a bijection function.
- Inverse of a bijection is unique.
- $(f^{-1})^{-1} = f$
- If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is a bijection and $(gof)^{-1} = (f^{-1}og^{-1})$
- $fog \neq gof$ but if $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$. Also, $(fog)(x) = (gof)(x) = (x)$.
- If $f: A \to B$ is a bijection then $f^{-1}: B \to A$ is an inverse function of f. $f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here I_A, is an identity function on set A and I_B is an identity function on set B.

PROBLEMS

Single Correct Answer Type

1. If $f(x) = \dfrac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \dfrac{1}{\sqrt{x - 2\sqrt{2x-4}}}$ for $x > 2$, then $f(11) =$

(a) 7/6 (b) 5/6 (c) 6/7 (d) 5/7

2. If for two functions g and f, gof is both injective and surjective, then which of the following is true?

(a) g and f should be injective and surjective
(b) g should be injective and surjective
(c) f should be injective and surjective
(d) none of these

3. The function $f: R \to R$, defined by $f(x) = x^2 + x$ is

(a) one-one onto (b) one-one into
(c) many-one onto (d) many-one into

4. If $f(x, y) = \dfrac{x}{y}$ where $(x, y) \in R$ and $x, y \neq 0$, then this function is a/an

(a) surjective (b) bijective
(c) one-one (d) none of these

5. If $f(x)$ is periodic function with period T then the function $f(ax + b)$ where $a > 0$, is periodic with period

(a) T/b (b) aT (c) bT (d) T/a

6. Domain of $f(x) = \log|\log x|$ is

(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(0, 1) \cup (1, \infty)$ (d) $(-\infty, 1)$

7. The domain of the function $f(x) = \sin^{-1}[\log_2(x/2)]$ is

(a) $[1, 4]$ (b) $[-4, 1]$
(c) $[-1, 4]$ (d) none of these

8. The domain of the function $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(a) $R - \{-1, -2\}$ (b) $(-2, \infty)$
(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

9. The domain of the function $f(x) = \sqrt{\log \dfrac{1}{|\sin x|}}$ is

(a) $R - \{2n\pi, n \in I\}$ (b) $R - \{n\pi, n \in I\}$
(c) $R - \{-\pi, \pi\}$ (d) $(-\infty, \infty)$

10. The domain of the function $f(x) = \log(\sqrt{x-4} + \sqrt{6-x})$ is

(a) $[4, \infty)$ (b) $(-\infty, 6]$
(c) $[4, 6]$ (d) none of these

11. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & , |x| \leq 1 \\ \dfrac{1}{2}(|x| - 1) & , |x| > 1 \end{cases}$$ is

(a) $R - \{0\}$ (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

12. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is

(a) $(-3, -1) \cup (1, \infty)$ (b) $[-3, -1] \cup [1, \infty)$
(c) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
(d) $[-3, -2) \cup (-2, -1) \cup [1, \infty)$

13. If n is an integer, then the domain of the function $f(x) = \sqrt{\sin 2x}$ is

(a) $\left[n\pi - \dfrac{\pi}{2}, n\pi\right]$ (b) $\left[n\pi, n\pi + \dfrac{\pi}{2}\right]$
(c) $[(2n-1)\pi, 2n\pi]$ (d) $[2n\pi, (2n+1)\pi]$

14. Domain of the function $\sqrt{\log\{(5x - x^2)/6\}}$ is

(a) (2, 3) (b) [2, 3] (c) [1, 2] (d) [1, 3]

15. The domain of the function $f(x) = \sin^{-1}\{(1 + e^x)^{-1}\}$ is

(a) (1/4, 1/3) (b) $[-1, 0]$
(c) $[0, 1]$ (d) $[-1, 1]$

16. The domain of the function $y = \dfrac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(-\infty, 0]$ (c) $(-\infty, -1)$ (d) $(-\infty, \infty)$

17. The function $f: R \to R$, $f(x) = x^2$ $\forall x \in R$ is

(a) injective but not surjective
(b) surjective but not injective
(c) injective as well as surjective
(d) neither injective nor surjective

18. If $f(x)=\frac{x}{1+x}$, then $f^{-1}(x)$ is equal to

(a) $\frac{(1+x)}{x}$ (b) $\frac{1}{(1+x)}$ (c) $\frac{(1+x)}{(1-x)}$ (d) $\frac{x}{(1-x)}$

19. The inverse of the function $\frac{10^x-10^{-x}}{10^x+10^{-x}}$ is

(a) $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{2}\log_{10}\left(\frac{1-x}{1+x}\right)$

(c) $\frac{1}{4}\log_{10}\left(\frac{2x}{2-x}\right)$ (d) none of these

20. If $f(x)=\frac{x}{\sqrt{1+x^2}}$, then $(fofof)(x)$ =

(a) $\frac{3x}{\sqrt{1+x^2}}$ (b) $\frac{x}{\sqrt{1+3x^2}}$

(c) $\frac{x}{\sqrt{1+x^2}}$ (d) none of these

21. If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(gof)(x)=2x^2-5x+2$, then $f(x)$ is

(a) $2x - 3$ (b) $2x + 3$

(c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

22. If from $R \to R$, $f(x) = (x + 1)^2$, $g(x) = x^2 + 1$, then $(fog)(-3)$ equals

(a) 121 (b) 112

(c) 211 (d) none of these

23. Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$, $c \neq 0$. Assume $a = 1$, $b = 2$. If $(fog)(x) = (gof)(x)$ for all x, then what can you say about c and d ?

(a) c and d both arbitrary (b) $c = 1$, d arbitrary

(c) c arbitrary, $d = 1$ (d) none of these

24. Let $g(x) = 1 + x - [x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0,\\ 1, & x>0\end{cases}$ then for all x, $f(g(x))$ is equal to

(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

25. If $f(x)=\frac{\alpha x}{x+1}, x\neq -1$. Then, value of α for which $f(f(x)) = x$ is

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

26. Let $f : (-1, 1) \to B$ be a function defined by $f(x)=\tan^{-1}\frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval

(a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $\left(0,\frac{\pi}{2}\right)$ (d) $\left[0,\frac{\pi}{2}\right)$

27. If $f(x) = 2x^6 + 3x^4 + 4x^2$, then $f'(x)$ is

(a) an even function

(b) an odd function

(c) neither even nor odd

(d) can't be determined

28. If a real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x) f(y) - f(a - x) f(a + y)$, where a is a given constant and $f(0) = 1$ then, $f(2a - x)$ is equal to

(a) $f(a) + f(a - x)$ (b) $f(-x)$

(c) $-f(x)$ (d) $f(x)$

29. If $g : [-2, 2] \to R$ where $g(x)=x^3+\tan x+\left[\frac{x^2+1}{P}\right]$ is an odd function where $[\cdot]$ denotes greatest integer function, then the value of parameter P is

(a) $-5 < P < 5$ (b) $P < 5$

(c) $P > 5$ (d) none of these

30. If $f(x)=\begin{cases}x, & \text{when } x \text{ is rational}\\ 0, & \text{when } x \text{ is irrational}\end{cases}$

and $g(x)=\begin{cases}0, & \text{when } x \text{ is rational}\\ x, & \text{when } x \text{ is irrational}\end{cases}$, then $(f - g)$ is

(a) one-one onto (b) one-one not onto

(c) not one-one but onto

(d) neither one-one nor onto

31. Range of the function $f : R \to R$ given by $f(x)=\frac{x^2}{x^2+1}$ is

(a) $(-1, 0)$ (b) $(-1, 1)$ (c) $[0, 1)$ (d) $(1, 1)$

32. The function f satisfies the functional equation $3f(x)+2f\left(\frac{x+59}{x-1}\right)=10x+30$ for all real $x \neq 1$. The value of $f(7)$ is

(a) 8 (b) 4 (c) -8 (d) 11

33. If $e^x = y+\sqrt{1+y^2}$, then y =

(a) $\frac{e^x+e^{-x}}{2}$ (b) $\frac{e^x-e^{-x}}{2}$

(c) $e^x + e^{-x}$ (d) $e^x - e^{-x}$

Multiple Correct Answer Type

34. If $f(x_1)-f(x_2)=f\left(\frac{x_1-x_2}{1-x_1x_2}\right)$ for $x_1, x_2 \in [-1, 1]$, then $f(x)$ is

(a) $\log\frac{(1-x)}{(1+x)}$ (b) $\tan^{-1}\frac{(1-x)}{(1+x)}$

(c) $\log\frac{(1+x)}{(1-x)}$ (d) $\tan^{-1}\frac{(1+x)}{(1-x)}$

35. Let $f(x)=\frac{\sin x}{\sqrt{1+\tan^2 x}}+\frac{\cos x}{\sqrt{1+\cot^2 x}}$, then

(a) fundamental period of $f(x)$ is 2π

(b) fundamental period of $f(x)$ is π

(c) domain of $f(x)$ is R

(d) range of $f(x)$ is $[-1, 1]$

36. Let $f : R \to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}, \forall x \in R$. Then which of the following statement(s) is/are true?

(a) $f(x + 2) = f(x)$ (b) $f(x + 4) = f(x)$

(c) $f(x + 6) = f(x)$ (d) $f(x + 8) = f(x)$

37. Let $R = \{(x, y) : x, y \in R, x^2 + y^2 \le 25\}$ and $R'=\left\{(x, y): x, y \in R, y \ge \frac{4}{9}x^2\right\}$. Then,

(a) domain of $R \cap R' = [-3, 3]$

(b) range of $R \cap R' = [0, 4]$

(c) range of $R \cap R' = [0, 5]$

(d) none of these

38. Let f be a function defined by $f(x)=\frac{x-5}{x-3}$, $x \ne 3, 2$; $f^k(x)$ denote the composition of f with itself taken k times *i.e.*, $f^3(x) = f(f(f(x)))$. Then,

(a) $f^{2012}(2009) = 2009$ (b) $f^{2009}(2010)=\frac{2006}{2007}$

(c) $f^{2009}(2011)=\frac{1003}{1004}$ (d) $f^{2012}(2012) = 2012$

Comprehension Type

Paragraph for Q. No. 39-41

A function $f(x)$ is said to be even iff $f(-x) = f(x)$ and odd iff $f(x) = -f(x)$. Also $f(x)$ is said to be periodic iff $f(x + T) = f(x)$, $T > 0$ and the least positive value T is called its period.

39. If $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$, then $f(x)$ is

(a) an even function (b) an odd function

(c) neither even nor odd

(d) both even and odd

40. $f(x)$ is an even polynomial function. Then $\sin(f(x) - 3x)$ is

(a) an even function (b) an odd function

(c) neither even nor odd

(d) periodic with period $\pi/2$

41. If $f(x)$ and $g(x)$ be two functions with all real numbers as their domains, then $h(x)=[f(x)+f(-x)][g(x)-g(-x)]$ is

(a) an even function when f is even and g is odd

(b) an even function when both f and g are odd

(c) always an odd function

(d) always an even function

Paragraph for Q. No. 42-44

Let us consider the two non empty sets X and Y such that $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $Y = \{1, 2, 3, 4, 5, 6\}$.

42. The number of into functions from X to Y are

(a) $7^6 - 3 \cdot 7!$ (b) $6^7 - 3 \cdot 7!$

(c) $6^7 - 7!$ (d) $6^6 - 3 \cdot 6!$

43. The number of many one functions from X to Y, is

(a) 6^7 (b) 7^6 (c) $7^6 - 6^7$ (d) $6^7 - 6!$

44. The number of functions from X to Y and such that $f(x_i) \ne x_i \ \forall \ x_i \in \{1, 2, 3, ... , 7\}$, is

(a) $6(5)^6$ (b) $5(6)^5$

(c) $5(7)^6$ (d) none of these

Matrix-Match Type

45. Match the following:

	Column- I		Column- II
A.	If f is a function such that $f(0) = 2$, $f(1) = 3$ and $f(x + 2) = 2f(x) - f(x + 1)$, then $f(5)$ is equal to	(p)	4
B.	If $f(x)=\begin{cases}x^2, & \text{for } x \ge 0\\ x, & \text{for } x<0\end{cases}$, then $f(\sqrt{13})=$	(q)	3
C.	If $f(x) + 2f(1 - x) = x^2 + 2$ for all $x \in R$, then $f(5)$ is	(r)	12
D.	If $f(x)=\frac{4^x}{4^x+2}$ for all $x \in R$, then $\sum_{k=1}^{6} f\left(\frac{k}{7}\right)$ equals	(s)	2
		(t)	13

46. Match the following:

Column-I		Column-II	
A.	Let $f: R \to R$ satisfies $f(x+y)+f(x-y)=2f(x)f(y)\ \forall\ x, y \in R$ and $f(0) \neq 0$, then $f(x)$ is	(p)	even
B.	Let $f: R \to R$ is defined by $f(x)=\frac{e^{\lvert x\rvert}-e^{-x}}{e^{x}+e^{-x}}$, then $f(x)$ is	(q)	odd
C.	Let $f: R \to R$ be a polynomial function satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(3)=28$, then $f(x)$ is	(r)	into
D.	Let $f: R \to R$ is defined by $f(x)=2x+\sin x$, then $f(x)$ is	(s)	many-one
		(t)	bijective

Integer Answer Type

47. If n is a natural number and $1 \le n \le 100$, then the number of solutions of $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{5}\right]=\frac{n}{2}+\frac{n}{3}+\frac{n}{5}$ (where [.] denotes the greatest integer function) is

48. If f is a function such that $f(x)+2f(1-x)=x^2+1$. then value of $f(3)$ is

49. The number of solutions of the equation $x^2+[x]-4x+3=0$ is where ([.] denotes the greatest integer function)

50. If the function $f(x)=\begin{cases} x+1, & x \le 1 \\ 2x+1, & 1<x\le 2 \end{cases}$ and $g(x)=\begin{cases} x^2, & -1 \le x \le 2 \\ x+2, & 2 \le x \le 3 \end{cases}$ then the number of roots of the equation $f(g(x))=2$ is

SOLUTIONS

1. (c) : $f(x)=\frac{1}{\sqrt{x+2\sqrt{2x-4}}}+\frac{1}{\sqrt{x-2\sqrt{2x-4}}}$

$\therefore\ f(11)=\frac{1}{\sqrt{11+2\sqrt{18}}}+\frac{1}{\sqrt{11-2\sqrt{18}}}$

$=\frac{1}{3+\sqrt{2}}+\frac{1}{3-\sqrt{2}}=\frac{3-\sqrt{2}}{7}+\frac{3+\sqrt{2}}{7}=\frac{6}{7}$

2. (a) : As gof is bijective, then g and f are also bijective.

3. (d) : $\because\ f(0)=f(-1)=0 \Rightarrow f(x)$ is many one. Also, there is no pre-image of -1. Hence, $f(x)$ is into function.

4. (a) : $\because\ f(\alpha x, \alpha y)=\frac{\alpha x}{\alpha y}=\frac{x}{y}(\alpha \neq 0)$.

Hence, $f(x, y)$ is not an injective function.

5. (d) : $f\left(a\left(x+\frac{T}{a}\right)+b\right)=f(ax+b+T)=f(ax+b)$.

6. (c) : $f(x)=\log|\log x|$, $f(x)$ is defined if $|\log x|>0$ and $x>0$.

i.e., $x>0$ and $x \neq 1$ $\quad(\because\ |\log x|>0, \text{ if } x \neq 1)$

$\therefore\ x \in (0,1) \cup (1, \infty)$

7. (a) : We have, $f(x)=\sin^{-1}[\log_2(x/2)]$.

Domain of $\sin^{-1}x$ is $[-1, 1]$

$\Rightarrow\ -1 \le \log_2(x/2) \le 1,\ \Rightarrow\ \frac{1}{2} \le \frac{x}{2} \le 2 \Rightarrow 1 \le x \le 4$

$\therefore\ x \in [1, 4]$.

8. (d) : Here, $x+3>0$ and $x^2+3x+2 \neq 0$

$\therefore\ x>-3$ and $(x+1)(x+2) \neq 0$, *i.e.*, $x \neq -1, -2$

$\therefore\ $ Domain $=(-3, \infty)-\{-1, -2\}$.

9. (b) : We have, $f(x)=\sqrt{\log\frac{1}{|\sin x|}}$

$\Rightarrow\ \sin x \neq 0 \Rightarrow x \neq n\pi, n \in I$

$\therefore\ $ Domain of $f(x)=R-\{n\pi, n \in I\}$.

10. (c) : $f(x)=\log(\sqrt{x-4}+\sqrt{6-x})$

$\Rightarrow\ \sqrt{x-4}+\sqrt{6-x}>0$

$\therefore\ $ Domain of $f(x)=[4, 6]$.

11. (c) : We have, $f(x)=\begin{cases} \frac{1}{2}(-x-1), & x<-1 \\ \tan^{-1}x, & -1 \le x \le 1; \\ \frac{1}{2}(x-1), & x>1 \end{cases}$

$\therefore\ f'(x)=\begin{cases} -\frac{1}{2}, & x<-1 \\ \frac{1}{1+x^2}, & -1 \le x \le 1 \\ \frac{1}{2}, & x>1 \end{cases}$

Now, $L[f'(-1)]=-\frac{1}{2}$ and $R[f'(-1)]=\frac{1}{1+(-1)^2}=\frac{1}{2}$

Also, $L[f'(1)]=\frac{1}{1+(1)^2}=\frac{1}{2}$; $R[f'(1)]=\frac{1}{2}$

$\because$ $f'(-1)$ does not exist

$\therefore$ Domain of $f(x) = R - \{-1\}$.

12. (c) : We have, $f(x) = \log_{3+x}(x^2 - 1)$

$\therefore$ $x^2 > 1 \Rightarrow x < -1$ or $x > 1$ and $3 + x > 0$

$\Rightarrow$ $x > -3$. Also, $x \neq -2$

$\therefore$ $D_f = (-3, -2) \cup (-2, -1) \cup (1, \infty)$.

13. (b) : $\because$ $\sqrt{\sin 2x}$ can't be negative

$\therefore$ Domain of function $\sqrt{\sin 2x}$ is $\left[n\pi, n\pi + \frac{\pi}{2}\right]$.

14. (b) : $\log\left\{\frac{5x - x^2}{6}\right\} \geq 0 \Rightarrow \frac{5x - x^2}{6} \geq 1$

or $x^2 - 5x + 6 \leq 0 \Rightarrow 2 \leq x \leq 3$

15. (a) : Clearly, $-1 \leq \frac{1}{1 + e^x} \leq 1$

But $2 < e^x < 3 \Rightarrow 3 < (e^x + 1) < 4 \Rightarrow \frac{1}{4} < \frac{1}{1 + e^x} < \frac{1}{3}$

$\therefore$ Domain of $f(x) = \left(\frac{1}{4}, \frac{1}{3}\right)$.

16. (a) : We have, $y = \frac{1}{\sqrt{|x| - x}}$

$\Rightarrow$ $|x| - x > 0$ or $|x| > x$

But $|x| = x$ for all positive x and $|x| > x$ for all negative x.

So, domain will be $(-\infty, 0)$.

17. (d) : Since the mapping is many-one into.

18. (d) : Let $y = f(x) = \frac{x}{1 + x} \Rightarrow x = f^{-1}(y)$

$\therefore$ $y = \frac{x}{1 + x} \Rightarrow y + yx = x \Rightarrow x = \frac{y}{1 - y}$

$\Rightarrow$ $f^{-1}(y) = \frac{y}{1 - y} \Rightarrow f^{-1}(x) = \frac{x}{1 - x}$.

19. (a) : Let $y = f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

$\Rightarrow$ $x = \frac{1}{2}\log_{10}\left(\frac{1 + y}{1 - y}\right)$

$\Rightarrow$ $f^{-1}(y) = \frac{1}{2}\log_{10}\left(\frac{1 + y}{1 - y}\right)$ ($\because y = f(x) \Rightarrow x = f^{-1}(y)$)

$\therefore$ $f^{-1}(x) = \frac{1}{2}\log_{10}\left(\frac{1 + x}{1 - x}\right)$.

20. (b) : $(fofof)(x) = (fof)(f(x)) = (fof)\left(\frac{x}{\sqrt{1 + x^2}}\right)$

$$= f\left[\frac{\left(\frac{x}{\sqrt{1 + x^2}}\right)}{\sqrt{1 + \frac{x^2}{1 + x^2}}}\right] = f\left(\frac{x\sqrt{1 + x^2}}{\sqrt{1 + x^2}\sqrt{1 + 2x^2}}\right)$$

$$= f\left(\frac{x}{\sqrt{1 + 2x^2}}\right) = \frac{\frac{x}{\sqrt{1 + 2x^2}}}{\sqrt{\left[1 + \frac{x^2}{1 + 2x^2}\right]}} = \frac{x}{\sqrt{1 + 3x^2}}.$$

21. (a) : Given, $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2$

or $\frac{1}{2}g[f(x)] = 2x^2 - 5x + 2$

$\therefore$ $[\{f(x)\}^2 + \{f(x)\} - 2] = 2[2x^2 - 5x + 2]$

$\Rightarrow$ $[f(x)]^2 + f(x) - (4x^2 - 10x + 6) = 0$

$\therefore$ $f(x) = \frac{-1 \pm \sqrt{1 + 4(4x^2 - 10x + 6)}}{2}$

$= \frac{-1 \pm \sqrt{(16x^2 - 40x + 25)}}{2} = \frac{-1 \pm (4x - 5)}{2} = 2x - 3, 2 - 2x$

22. (a) : $f(x) = (1 + x)^2$ and $g(x) = x^2 + 1$

Also, $fog(-3) = f[g(-3)] = f[9 + 1] = f[10] = [11]^2 = 121$.

23. (b) : $(fog)(x) = f(g(x)) = a(cx + d) + b$

and $(gof)(x) = g(f(x)) = c(ax + b) + d$

Given that, $(fog)(x) = (gof)(x)$

At $a = 1$, $b = 2$, we have

$cx + d + 2 = cx + 2c + d$

$\Rightarrow$ $c = 1$ and d is arbitrary.

24. (b) : Here $g(x) = 1 + n - n = 1$, $x = n \in Z$

$g(x) = 1 + n + k - n = 1 + k$, $x = n + k$

(where $n \in Z$, $0 < k < 1$)

Clearly, $g(x) > 0$ for all x. So, $f(g(x)) = 1$ for all x.

25. (d) : $f(f(x)) = \frac{\alpha f(x)}{f(x) + 1} = \frac{\alpha\left(\frac{\alpha x}{x + 1}\right)}{\left(\frac{\alpha x}{x + 1} + 1\right)} = \frac{\alpha^2 \cdot x}{\alpha x + x + 1}$

Now, $x = \frac{\alpha^2 \cdot x}{(\alpha + 1)x + 1} \Rightarrow x[(\alpha + 1)x + 1 - \alpha^2)] = 0$

or $(\alpha + 1)x^2 + (1 - \alpha^2)x = 0$.

This should hold for all x.

$\Rightarrow$ $\alpha + 1 = 0$, $1 - \alpha^2 = 0 \Rightarrow \alpha = -1$.

26. (b) : For $-1<x<1, \tan^{-1}\dfrac{2x}{1-x^2}=2\tan^{-1}x$

Range of $f(x)=\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\therefore$ Co-domain of function $=B=\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

27. (b) : We have, $f(x) = 2x^6 + 3x^4 + 4x^2$

$f(-x) = 2(-x)^6 + 3(-x)^4 + 4(-x)^2 = f(x)$

$\therefore$ $f(x)$ is an even function and derivative of an even function is always odd.

28. (c) : $f(a - (x - a)) = f(a)f(x - a) - f(0)f(x)$ ...(i)

Put $x = 0$, $y = 0$; $f(0) = (f(0))^2 - [f(a)]^2 \Rightarrow f(a) = 0$

From (i), $f(2a - x) = -f(x)$. $[\because f(0) = 1]$

29. (c) : Given, $g(x)=x^3+\tan x+\left[\dfrac{x^2+1}{P}\right]$

$\therefore$ $g(-x)=-x^3-\tan x+\left[\dfrac{x^2+1}{P}\right]$

Now, $g(x) + g(-x) = 0$ $[\because$ $g(x)$ is an odd function]

$\therefore$ $x^3+\tan x+\left[\dfrac{x^2+1}{P}\right]-x^3-\tan x+\left[\dfrac{x^2+1}{P}\right]=0$

$\Rightarrow$ $2\left[\dfrac{x^2+1}{P}\right]=0 \Rightarrow 0\le\dfrac{x^2+1}{P}<1$

Since, $x \in [-2, 2] \Rightarrow 0\le\dfrac{5}{P}<1 \Rightarrow P>5$.

30. (a) : $(f-g)(x)=\begin{cases} x, & x\in Q \\ -x, & x\notin Q \end{cases}$

31. (c) : Let $y=\dfrac{x^2}{x^2+1}$

$\Rightarrow$ $(y - 1)x^2 + 0x + y = 1$, $y \neq 1$

For real value of x, $D \ge 0$

$\Rightarrow -4y(y - 1) \ge 0 \Rightarrow y(y - 1) \le 0$

$\Rightarrow$ $y \le [0, 1)$ $\left[\because 0\le\dfrac{x^2}{x^2+1}<1\right]$

32. (b) : We have, $3f(x)+2f\left(\dfrac{x+59}{x-1}\right)=10x+30$

For $x = 7$, $3f(7) + 2f(11) = 70 + 30 = 100$

For $x = 11$, $3f(11) + 2f(7) = 140$

$\therefore$ $\dfrac{f(7)}{-20}=\dfrac{f(11)}{-220}=\dfrac{-1}{9-4} \Rightarrow f(7)=4.$

33. (b) : $\because$ $e^x=y+\sqrt{1+y^2}$ $\therefore$ $e^x-y=\sqrt{1+y^2}$

Squaring both the sides, $(e^x - y)^2 = (1 + y^2)$

$e^{2x} + y^2 - 2ye^x = 1 + y^2 \Rightarrow e^{2x} - 1 = 2ye^x$

$\Rightarrow$ $2y=\dfrac{e^{2x}-1}{e^x} \Rightarrow y=\dfrac{e^x-e^{-x}}{2}$

34. (a,b,c) : When $x_1 = -1$ and $x_2 = 1$, then

$f(-1)-f(1)=f\left[\dfrac{-1-1}{1+1(1)}\right]=f(-1)\Rightarrow f(1)=0,$

which is satisfied when $f(x)=\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

When $x_1 = x_2 = 0$, then

$f(0)-f(0)=f\left[\dfrac{0-0}{1-0}\right]=f(0)\Rightarrow f(0)=0$ ...(i)

Also, when $x_1 = -1$ and $x_2 = 0$ then

$f(-1)-f(0)=f\left(\dfrac{-1-0}{1-0}\right)=f(-1)\Rightarrow f(0)=0$ ...(ii)

Both (i) and (ii) are satisfied when

$f(x)=\log\left(\dfrac{1-x}{1+x}\right)$ and $f(x)=\log\left(\dfrac{1+x}{1-x}\right)$

35. (a,d) : $f(x)=\begin{cases} \sin 2x, & 0<x<\dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2}<x<\pi \\ -\sin 2x, & \pi<x<\dfrac{3\pi}{2} \\ 0, & \dfrac{3\pi}{2}<x<2\pi \end{cases}$

So, $f(x)$ is periodic with fundamental period 2π and range $[-1, 1]$.

36. (b, d) : Given, $f(x+1)=\dfrac{f(x)-5}{f(x)-3}$ ...(i)

$\Rightarrow$ $f(x)=\dfrac{3f(x+1)-5}{f(x+1)-1}$

Replace 'x' by $x - 1$, $f(x-1)=\dfrac{3f(x)-5}{f(x)-1}$ ...(ii)

Replace 'x' by $x + 1$ in (i)

$f(x+2)=\dfrac{f(x+1)-5}{f(x+1)-3}=\dfrac{2f(x)-5}{f(x)-2}$ ...(iii)

Similarly, $f(x-2)=\dfrac{2f(x)-5}{f(x)-2}$

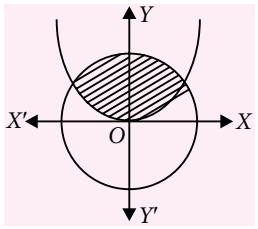
$\therefore$ $f(x + 2) = f(x - 2) \Rightarrow f(x + 4) = f(x)$

and $f(x + 8) = f(x + 4) = f(x)$

37. (a, c) : We have, $x^2 + y^2 \le 25$

$\Rightarrow \quad 9y \ge 4x^2 \Rightarrow x^2 \le \frac{9}{4}y$

$x^2 = \frac{9y}{4}$



Now, $\frac{9y}{4} + y^2 - 25 = 0$

$\Rightarrow \quad 4y^2 + 9y - 100 = 0$

$\Rightarrow \quad (y-4)\left(y+\frac{50}{8}\right)=0$

$\Rightarrow \quad y = 4$

Domain of $R \cap R' = [-3, 3]$

Range of $R \cap R' = [0, 5]$

38. (a, c, d) : $f(x)=\frac{x-5}{x-3} \Rightarrow f^2(x)=\frac{2x-5}{x-2}=g(x)$

Then, $g^{2n}(x) = x$ and $g^{2n+1} = g(x)$

$f^{2012}(2009) = g^{1006}(2009) = 2009$

$f^{2009}(2010)=f[g^{1004}(2010)]=f(2010)=\frac{2005}{2007}$

$f^{2009}(2011)=f[g^{1004}(2011)]=f(2011)=\frac{2006}{2008}$

$f^{2012}(2012) = g^{1006}(2012) = 2012$

39. (a) : $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$

$f(-x)=\frac{-x}{e^{-x}-1}-\frac{x}{2}+1=\frac{xe^x}{e^x-1}-\frac{x}{2}+1=\frac{x}{e^x-1}+\frac{x}{2}+1$

$= f(x)$

40. (c) : Let $g(x) = \sin[f(x) - 3x]$

$g(-x) = \sin[f(-x) + 3x] = \sin[f(x) + 3x]$

41. (c) : $h(-x) = [f(-x) + f(x)][g(-x) - g(x)] = -h(x)$

42. (b) : Number of into functions $= 6^7 - 3 \cdot 7!$

43. (a) : Number of many-one functions $= 6^7$.

44. (a) : 7 can be associated to $y \in Y$ in 6 ways and 1, 2, 3, 4, 5, 6 can be associated in 5 ways each.

$\therefore$ The total number of functions required $= 6(5)^6$.

45. A → (t); B → (t); C → (q); D → (q)

A. Given $f(0) = 2, f(1) = 3$

and $f(x + 2) = 2f(x) - f(x + 1)$

Put $x = 0$; $f(2) = 2f(0) - f(1) = 2 \times 2 - 3 = 1$

Put $x = 1$; $f(3) = 2f(1) - f(2) = 2 \times 3 - 1 = 5$

Put $x = 2$; $f(4) = 2f(2) - f(3) = 2 \times 1 - 5 = -3$

Put $x = 3$; $f(5) = 2f(3) - f(4) = 2 \times 5 - (-3) = 13$

B. $\sqrt{13} > 0 \Rightarrow f\left(\sqrt{13}\right)=\left(\sqrt{13}\right)^2=13$

C. Given $f(x)+2f(1-x)=x^2+2, \forall x \in R$ ...(i)

Replace x by $1 - x$, we get

$f(1-x)+2f(x)=(1-x)^2+2$ ...(ii)

Solving (i) & (ii), we get $f(x)=\frac{(x-2)^2}{3}$

$\therefore \quad f(5) = 3.$

D. Given, $f(x)=\frac{4^x}{4^x+2}$

$\Rightarrow f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}=\frac{4}{4+2\cdot 4^x}=\frac{2}{4^x+2}$

$\therefore \quad f(x) + f(1 - x) = 1$

$\Rightarrow f\left(\frac{1}{7}\right)+f\left(\frac{6}{7}\right)=1; \; f\left(\frac{2}{7}\right)+f\left(\frac{5}{7}\right)=1$

and $f\left(\frac{3}{7}\right)+f\left(\frac{4}{7}\right)=1 \Rightarrow \sum_{k=1}^{6} f\left(\frac{k}{7}\right)=1+1+1=3$

46. A → (p, r, s) ; B → (r,s) ; C → (t) ; D → (q, t)

A. Put $x = y = 0$, then $2f(0) = 2\{f(0)\}^2 \Rightarrow f(0) = 1$

Now put $x = 0$, then

$f(y) + f(-y) = 2f(0) f(y)$

$\therefore \quad f(y) = f(-y) \Rightarrow f(x)$ is even function

$\Rightarrow \quad f(x)$ is many one and cannot be onto.

B. If $x \le 0$ then $f(x) = 0 \Rightarrow f(x)$ is many one and neither even nor odd. Clearly, $f(x)$ is not onto.

C. $f(x)$ is either $-x^n + 1$ or $x^n + 1$. But $f(3) = 28$

$\therefore \quad f(x) = x^3 + 1$ which is neither even nor odd but one-one and onto both.

D. Obviously, $f(x)$ is odd and $f'(x) = 2 + \cos x > 0$, so $f(x)$ is one-one.

Also, $f(x) \to \infty$ if $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$ so $f(x)$ is onto.

47. (3) : From the given equation, we have

$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$

$\Rightarrow \left\{\frac{n}{2}\right\}+\left\{\frac{n}{3}\right\}+\left\{\frac{n}{5}\right\}=0$; where $\{.\}$ is a fractional part .

MPP-5 CLASS XI ANSWER KEY

1.	(d)	**2.**	(a)	**3.**	(c)	**4.**	(b)	**5.**	(a)
6.	(b)	**7.**	(c,d)	**8.**	(c)	**9.**	(a,b,c)	**10.**	(c,d)
11.	(b)	**12.**	(a,b,c)	**13.**	(b,c,d)	**14.**	(b)	**15.**	(b)
16.	(c)	**17.**	(6)	**18.**	(9)	**19.**	(3)	**20.**	(6)

But each of the fractional part function is positive and their sum is zero. Hence, each of the fractional part function is zero. Consequently, each of $\frac{n}{2}, \frac{n}{3}, \frac{n}{5}$ is an integer. The L.C.M. of 2, 3, 5 is 30. Therefore we can take $n = 30k$, where k is an integer.
Hence, the number of solutions such that $1 \le n \le 100$ is 3 (n = 30, 60 and 90).

48. (0) : $f(x) + 2f(1 - x) = x^2 + 1$ ...(i)
$\therefore \quad f(1 - x) + 2f(x) = (1 - x)^2 + 1 = x^2 - 2x + 2$ ...(ii)
Solving (i) and (ii), we get $3f(x) = x^2 - 4x + 3$
$\therefore \quad f(3) = 0$

49. (0) : From the given equation, we get
$x^2 + (x - f) - 4x + 3 = 0$, where f = fractional part function such that $0 \le f < 1$
$\Rightarrow \quad (x^2 - 3x + 3) - f = 0 \Rightarrow f = x^2 - 3x + 3$.
But $0 \le f < 1$. Therefore $0 \le x^2 - 3x + 3 < 1$ ...(i)
Now, $x^2 - 3x + 3 = 0$ gives $x = \frac{3 \pm \sqrt{9-12}}{2}$ (imaginary)
Again from $x^2 - 3x + 3 < 1$, we get $x^2 - 3x + 2 < 0$
$\Rightarrow \quad (x - 2)(x - 1) < 0 \Rightarrow 1 < x < 2$.
Thus the inequality (i) is satisfied if $1 < x < 2$
$\Rightarrow \quad [x] = 1$.
Putting $[x] = 1$ in the given equation, we get
$x^2 + 1 - 4x + 3 = 0 \Rightarrow (x - 2)^2 = 0 \Rightarrow x = 2$
But $x = 2$ does not satisfy the inequality $1 < x < 2$.
$\therefore \quad$ The given equation has no solution.

50. (2) : $f(g(x)) = \begin{cases} x^2 + 1, & -1 \le x \le 1 \\ 2x^2 + 1, & 1 < x \le \sqrt{2} \end{cases}$
Now, $f(g(x)) = 2 \Rightarrow x^2 + 1 = 2$ and $2x^2 + 1 = 2$
$\Rightarrow \quad x = \pm 1$ and $x = \pm \frac{1}{\sqrt{2}}$
$\therefore \quad$ Number of solutions = 2.

Four Mathematicians received the Prestigious Fields Medals, also known as the Nobel Prize of Mathematics

The Fields Medal is awarded by the International Mathematical Union once every four years, recognising outstanding Mathematical achievements of scholars under the age of 40. This award is often described as the Nobel Prize of Mathematics.

Akshay Venkatesh, a renowned Indian-Australian Mathematician, is one of four winners of Mathematics' prestigious Fields Medal. The other three winners are: Caucher Birkar, a Cambridge University professor of Iranian Kurdish origin; Germany's Peter Scholze, who teaches at the University of Bonn and Alessio Figalli, an Italian Mathematician at ETH Zurich.

Akshay Venkatesh has been awarded the medal this year for his work on using dynamic models to solve problems in number theory. "Just manipulating numbers makes me feel happy," he says. The Stanford Professor and father of two says he feels lucky to be a part of something so meaningful and continues to impress by detecting connections between diverse areas of Mathematics.

Caucher Birkar, the Cambridge Professor devotes his time to solving the problems of modern Mathematics, such as the minimal model and Fano varieties. " I am hoping this news will put a little smile on the lips of those 40 million people," Birkar said in his award reception video, referring to the Kurdish people.

Peter Scholze was the crowd favourite for this years medal, already considered one of the most influential Mathematicians in the world. The soft spoken specialist in arithmetic algebraic geometry is eager to continue his brilliant trajectory: "There are an infinite number of problems. As soon as you solve one problem, there are 10 more coming."

Alessio Figalli took home the prize for his contributions to optimal transportation theory. He says that he is happy that he has been able to solve some important problems in his career, but has work cut out for him for the next 30 or 40 years.

CONCEPT MAP

CIRCLES

Class XI Class XII

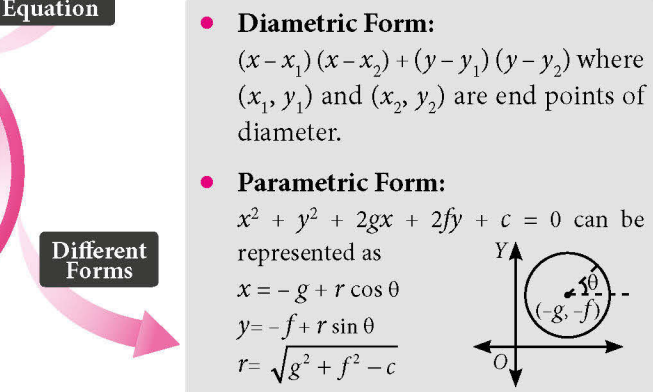
Definition

- A circle is defined as locus of all such points in plane which remains at constant distance from a given fixed point. Fixed point is called centre and the fixed distance (distance from centre to each point of locus) is called radius of the circle.
- The second degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents the equation of circle, if $a = b$, $h = 0$ and

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0 \text{ and } g^2 + f^2 - ac > 0$$

Equation

- General equation of circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(C) \equiv (-g, -f)$ and radius $(r) = \sqrt{g^2 + f^2 - c}$
- $(x - \alpha)^2 + (y - \beta)^2 = a^2$ with centre $(C) = (\alpha, \beta)$ and radius $(r) = a$
- Simplest | Touching x-axis | Touching y-axis | Touching both x & y-axis



Different Forms

- **Diametric Form:** $(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$ where (x_1, y_1) and (x_2, y_2) are end points of diameter.
- **Parametric Form:** $x^2 + y^2 + 2gx + 2fy + c = 0$ can be represented as

$x = -g + r\cos\theta$

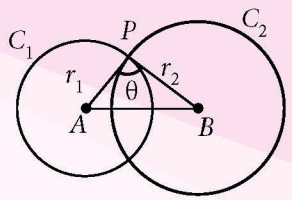
$y = -f + r\sin\theta$

$r = \sqrt{g^2 + f^2 - c}$

Angle of intersection between two circles

If the two circles C_1 and C_2 given by $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively intersect at a point P and PA, PB are the tangents at P, then the angle between the tangents at P (*i.e.* $\angle BPA$) is the angle of intersection of the two circles given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - (AB)^2}{2r_1r_2}$$



Orthogonal Circles

Two Circles are orthogonal if $\theta = 90°$

i.e., $r_1^2 + r_2^2 - (AB)^2 = 0$

or $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Important Results

- Let $S = x^2 + y^2 + 2gx + 2fy + c$

  $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
  - Equation of chord of the circle $S = 0$ whose mid-point is (x_1, y_1) is $T = S_1$.
  - Length of intercept made by the circle $S = 0$ with x-axis $= 2\sqrt{g^2 - c}$.
  - Equation of tangent to the circle $S = 0$ is given by $T = 0$.
  - Length of intercept made by the circle $S = 0$ with y-axis $= 2\sqrt{f^2 - c}$.
  - Length of tangent from a point (x_1, y_1) to the circle $S = 0$ is given by $\sqrt{S_1}$.
  - The equation of radical axis of two circles $S_1 = 0$, $S_2 = 0$ in which the coefficient of x^2, y^2 are same is the equation of the line $S_1 - S_2 = 0$.
- Line $y = mx + c$ will touch the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$.
- Equation of director circle of the circle $x^2 + y^2 = a^2$ is the circle whose radius is $\sqrt{2}$ times the radius of the original circle.

DIFFERENTIABILITY

Definition

A real valued function $f(x)$ is differentiable at $x = c$ if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely or $Lf'(a) = Rf'(a)$

where $Lf'(a)$ = Left hand derivative $= \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$ and

$Rf'(a)$ = Right hand derivative $= \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$

Note : If a function is differentiable at a point then it is continuous at that point. But the converse is not always true.

Some Standard Derivatives

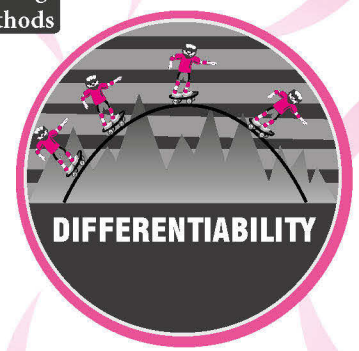
- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$
- $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1 - x^2}}$
- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$
- $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1 + x^2}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2 - 1}}$
- $\frac{d}{dx}a^x = a^x \log a$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}\ln(x + \sqrt{x^2 + 1}) = \frac{1}{\sqrt{x^2 + 1}}$
- $\frac{d}{dx}\ln(x + \sqrt{x^2 - 1}) = \frac{1}{\sqrt{x^2 - 1}}$

Solving Methods

- **Parametric Differentiation:** If $x = f(t)$ and $y = g(t)$ then, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
- **Logarithmic Differentiation:** If $y = u^v$, where u and v are functions of x, then apply log on both sides to get $\log y = v \log u$

$$\Rightarrow \frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$$

- **Chain Rule:** If $y = f(t)$, $t = g(u)$ and $u = m(x)$ then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$



Second Order Derivative

Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.

If $f'(x)$ is differentiable, then

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$

or $\frac{d^2y}{dx^2} = f''(x)$.

Formulae for Higher Order Derivatives

- $\frac{d^n}{dx^n}\ln(ax + b) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax + b)^n}$
- $\frac{d^n}{dx^n}\cos(ax + b) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$
- $\frac{d^n}{dx^n}\sin(ax + b) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$
- $\frac{d^n}{dx^n}e^{ax}\sin(bx + c) = r^n e^{ax}\sin(bx + c + n\theta)$
- $\frac{d^n}{dx^n}e^{ax}\cos(bx + c) = r^n e^{ax}\cos(bx + c + n\theta)$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\frac{b}{a}$

Leibnitz Formula

The n^{th} derivative of the product of two functions: If u, v be two functions possessing derivative of the n^{th} order, then

$(uv)_n = u_nv + {}^nC_1\,u_{n-1}v_1 + {}^nC_2\,u_{n-2}v_2 + \ldots + {}^nC_r\,u_{n-r}v_r + \ldots + {}^nC_n uv_n$

$= \sum_{r=0}^{n} C(n,r)\,u_{n-r}v_r$ where

$C(n,r) = {}^nC_r = \frac{n!}{(n-r)!r!}$

and C_0, C_1, ..., C_n are the binomial coefficients.

CLASS XII | Series 5

CBSE DRILL

Synopsis and Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2018-19.

Integrals | Application of Integrals

Integrals

INDEFINITE INTEGRAL

Let $F(x)$ be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$.

Then, $F(x)$ is called integral of $f(x)$. Symbolically, it is written as $\int f(x)\,dx = F(x) + C$ where C is the constant of integration. Here, $F(x)$ is also called anti-derivative or primitive function of $f(x)$.

Note: The indefinite integral of a function represents geometrically a family of curves having parallel tangents at their points of intersection with the lines orthogonal to the axis representing the variable of integration.

SOME FUNDAMENTAL INTEGRAL FORMULAE

• $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	• $\int \frac{1}{x} dx = \log\lvert x\rvert + C$	• $\int e^x dx = e^x + C$
• $\int a^x dx = \frac{a^x}{\log a} + C$	• $\int \sin x\, dx = -\cos x + C$	• $\int \cos x\, dx = \sin x + C$
• $\int \sec^2 x\, dx = \tan x + C$	• $\int \text{cosec}^2 x\, dx = -\cot x + C$	• $\int \sec x \tan x\, dx = \sec x + C$
• $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$	• $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	• $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$
• $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	• $\int -\frac{1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$	• $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
• $\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$		

PROPERTIES OF INTEGRATION

For any two functions $f(x)$ and $g(x)$, we have

(1) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

(2) For any constant k, $\int k \cdot f(x)dx = k\int f(x)dx$

METHODS OF INTEGRATION

There are three methods for finding integral of a function.

(i) Integration by Substitution

Let $I = \int f(x)dx$. Put $x = g(t)$ then $dx = g'(t)\,dt$. Thus $I = \int f(x)dx = \int f(g(t))g'(t)\,dt$.

Some important substitutions

Integral	Substitution
$\int f(ax+b)dx$	Put $ax + b = t$
$\int x^{n-1} f(x^n)dx$	Put $x^n = t$
$\int \{f(x)\}^n f'(x)dx$ or $\int \frac{f'(x)}{f(x)}dx$	Put $f(x) = t$
$\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Transformed into standard form by expressing $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$ then put $x+\frac{b}{2a} = t$
$\int (ax+b)\sqrt{cx+d}\,dx$ or $\int \frac{ax+b}{\sqrt{cx+d}}dx$	Put $ax + b = A(cx + d) + B$
$\int \frac{dx+e}{ax^2+bx+c}dx$ or $\int \frac{dx+e}{\sqrt{ax^2+bx+c}}dx$ or $\int (dx+e)\sqrt{ax^2+bx+c}\,dx$	Put $(dx+e) = A\frac{d}{dx}(ax^2+bx+c) + B$
$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$	Put $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$
$\int \sin^m x \cos^n x\,dx,\ m, n \in N$	• If m is odd and n is even put $\cos x = t$. • If n is odd and m is even put $\sin x = t$. • If both m, n are odd, put either $\sin x = t$ or $\cos x = t$. • If both m, n are even, then simplify the integral as sum of sines and cosines of multiple angles.

(ii) Integration by Partial Fractions

The following table gives us an idea of the types of partial fractions to be taken for different types of proper rational algebraic functions.

Rational form	Partial form
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)^3(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$ where $x^2 + bx + c$ cannot be factorised further

(iii) Integration by parts (Using ILATE Rule)

$$\int f(x).g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx$$

Here, functions $f(x)$ and $g(x)$ are chosen on the basis of ILATE where

I stands for Inverse Trigonometric Function
L stands for Logarithmic Function
A stands for Algebraic Function
T stands for Trigonometric Function
E stands for Exponential Function.

SOME PARTICULAR INTEGRALS

(i) $\int \cot x\, dx = \log|\sin x| + C$

(ii) $\int \tan x\, dx = \log|\sec x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(iv) $\int \text{cosec } x\, dx = \log|\text{cosec } x - \cot x| + C$

(v) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(vii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(viii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(ix) $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + C$

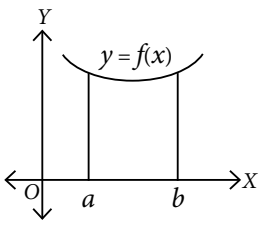
(x) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(xi) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(xii) $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

DEFINITE INTEGRAL

If we construct the graph of the integrand $y = f(x)$, then in the case $f(x) \geq 0$, the integral will be numerically equal to the area bounded by the curve $y = f(x)$, the axis of x and the straight lines $x = a$ and $x = b$.



Also, $\int_a^b f(x)dx = |F(x)|_a^b = F(b) - F(a)$, where $F(x)$ is an anti derivative of $f(x)$. If $f(x) \geq g(x)$ on $[a, b]$, then the area bounded by $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is $\int_a^b [f(x) - g(x)]dx$.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let $f(x)$ be a continuous function defined in a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. For this, interval $[a, b]$ is divided into n equal parts each of width h by inserting $(n - 1)$ points $a + h$, $a + 2h$, ..., $a + (n - 1)\,h$.

Then, $nh = b - a$. Also, $h \to 0$ as $n \to \infty$

$$\therefore \int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right]$$

FUNDAMENTAL THEOREM OF CALCULUS

First fundamental theorem : Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function, then $A'(x) = f(x)$, for all $x \in [a, b]$.

Second fundamental theorem : Let $f(x)$ be a continuous function defined on $[a, b]$ and $F(x)$ be the integral of $f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $z = \phi(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $\phi(a)$ and $\phi(b)$ respectively.

PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx = \int_a^b f(y)dy$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$. In particular, $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b$

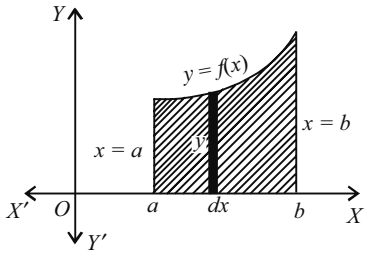
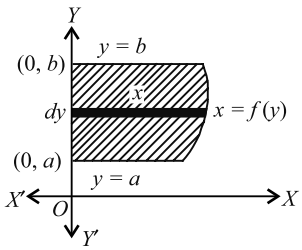
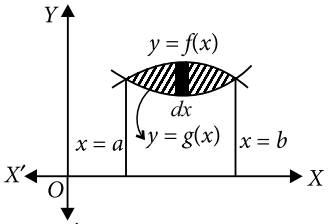
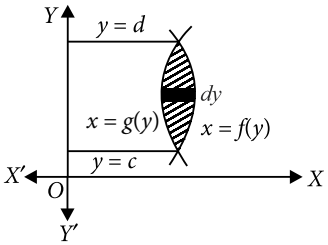
(iv) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(v) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

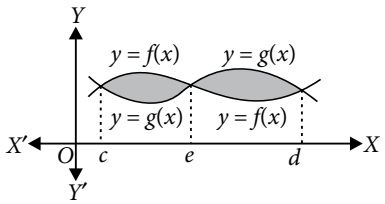
(vi) $\int_0^{2a} f(x)dx = \begin{cases} 0 & , \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & , \text{ if } f(2a-x) = f(x) \end{cases}$

(vii) $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & , \text{ if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx & , \text{ if } f(x) \text{ is an even function} \end{cases}$

Application of Integrals

	Explanation	Geometrical Representation
Area under simple curves	Let $f(x)$ be a continuous function defined on $[a, b]$, then the area bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ is given by $\int_a^b f(x)dx = \int_a^b ydx$	
	Area bounded by the curve $x = f(y)$, y-axis *i.e.* $x = 0$ and $y = a$ and $y = b$ is given by $\int_a^b f(y)dy = \int_a^b xdy$	
Area between two curves	The area bounded by the two curves $y = f(x)$, $y = g(x)$ is obtained by $A = \left\lvert \int_a^b f(x)dx - \int_a^b g(x)dx \right\rvert$ where $(a, f(a))$ and $(b, f(b))$ are intersecting points of $y = f(x)$ and $y = g(x)$.	
	The area bounded by the two curves $x = f(y)$, $x = g(y)$ is obtained by $A = \left\lvert \int_c^d f(y)dy - \int_c^d g(y)dy \right\rvert$ where $(f(c), c)$ and $(f(d), d)$ are intersecting points of $x = f(y)$ and $x = g(y)$	

Note : If $f(x) \geq g(x)$ in $[c, e]$ and $f(x) \leq g(x)$ in $[e, d]$ where $c < e < d$, then area bounded by the curves is given by $\int_c^e [f(x) - g(x)]\,dx + \int_e^d [g(x) - f(x)]\,dx$



WORK IT OUT

VERY SHORT ANSWER TYPE

1. Evaluate $\int \frac{4x-3}{\sqrt[3]{2x^2-3x+7}}dx$

2. Solve $\int 9^{\log_3 x}dx$

3. Find the value of $\int_1^2 |x-3|dx$.

4. Solve $\int \sec^4 x \tan x\,dx$

5. Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx$

SHORT ANSWER TYPE

6. Find the value of $\int e^{-x}\,\text{cosec}^2(2e^{-x}+5)dx$.

7. Show that $\int_0^1 \log\left(\frac{1}{x}-1\right)dx = 0$.

8. Evaluate $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}dx$

9. Using integration, find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 3$ and $x = 5$.

10. Evaluate $\int_0^1 \frac{xe^x}{(x+1)^2}dx$

LONG ANSWER TYPE - I

11. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

12. Find the area bounded by the curves $y = \cos x$ and $y = \sin x$ between the lines $x = 0$ and $x = 3\pi/2$.

13. Find $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

14. Find $\int (2x+3)\sqrt{x^2+4x+3}dx$

15. Find the area of the region bounded by the curve $y = x^2 - 6x + 5$, the x-axis and the lines $x = 2$, $x = 4$.

LONG ANSWER TYPE - II

16. Using integration, find the area of the triangle ABC whose vertices are $A(-1, 1)$, $B(0, 5)$ and C(3, 2).

17. Evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}dx$

18. Evaluate the following integrals:

(i) $\int \sec^{-1}\sqrt{x}\,dx$ (ii) $\int (1+x^2)\sin 3x\,dx$

19. Evaluate $\int_0^3 (x^2-2x+2)\,dx$ as the limit of a sum.

20. Evaluate $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$

SOLUTIONS

1. Put $2x^2 - 3x + 7 = t \Rightarrow (4x - 3)\,dx = dt$,

$\therefore \int \frac{4x-3}{\sqrt[3]{2x^2-3x+7}}dx = \int \frac{1}{t^{1/3}}dt = \int t^{-1/3}dt = \frac{t^{2/3}}{\frac{2}{3}} + C$

$= \frac{3}{2}(2x^2 - 3x + 7)^{2/3} + C.$

2. We have, $\int 9^{\log_3 x}dx = \int (3^2)^{\log_3 x}dx = \int 3^{2\log_3 x}dx$

$= \int 3^{\log_3 x^2}dx = \int x^2 dx = \frac{1}{3}x^3 + C.$

3. We have, $\int_1^2 |x-3|dx = \int_1^2 -(x-3)dx$

[$\because$ $1 < x < 2$ $\therefore$ $x - 3 < 0$]

$= -\left[\frac{x^2}{2} - 3x\right]_1^2 = -\left[\left(\frac{2^2}{2} - 3\times 2\right) - \left(\frac{1}{2} - 3\right)\right] = \frac{3}{2}.$

4. Put $\sec x = t \Rightarrow \sec x \tan x\,dx = dt$

$\therefore \int \sec^4 x \tan x\,dx = \int (\sec x)^3 \sec x \tan x\,dx = \int t^3\,dt$

$= \frac{t^4}{4} + C = \frac{1}{4}\sec^4 x + C.$

5. Let $f(x) = x^3 \sin^4 x$

Then, $f(-x) = (-x)^3 \sin^4(-x) = -x^3(-\sin x)^4 = -f(x)$

$\because$ $f(x)$ is an odd function $\therefore \int_{-\pi/4}^{\pi/4} f(x)dx = 0$.

6. Let $I = \int e^{-x}\,\text{cosec}^2(2e^{-x}+5)dx$

Put $2e^{-x} + 5 = t$

$\therefore 2e^{-x}(-1)dx = dt \Rightarrow e^{-x}dx = -\frac{1}{2}dt$

$\therefore I = \int \text{cosec}^2 t \cdot \left(-\frac{1}{2}dt\right) = -\frac{1}{2}(-\cot t) + C$

$= \frac{1}{2}\cot(2e^{-x} + 5) + C.$

7. Let $I = \int_0^1 f(x)dx$, where

$f(x) = \log\left(\frac{1}{x} - 1\right) = \log\left(\frac{1-x}{x}\right)$

Then, $f(1 - x) = \log\left(\frac{1-(1-x)}{1-x}\right) = \log\frac{x}{1-x}$

$= \log\left[\left(\frac{1-x}{x}\right)^{-1}\right] = -\log\left(\frac{1-x}{x}\right) = -f(x)$

$\because$ $f(1 - x) = -f(x)$ $\therefore \int_0^1 f(x)dx = 0$.

8. We have, $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}dx$

$= \int\left(\frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} \times \frac{\sqrt{2x+3}-\sqrt{2x-3}}{\sqrt{2x+3}-\sqrt{2x-3}}\right)dx$

$= \int \frac{\sqrt{2x+3}-\sqrt{2x-3}}{(2x+3)-(2x-3)}dx = \frac{1}{6}\int((2x+3)^{1/2}-(2x-3)^{1/2})dx$

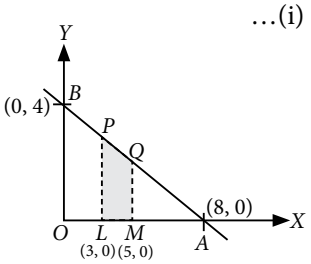
$= \frac{1}{6}\left(\frac{(2x+3)^{3/2}}{\frac{3}{2}.2} - \frac{(2x-3)^{3/2}}{\frac{3}{2}.2}\right)+C$

$= \frac{1}{18}[(2x+3)^{3/2} - (2x-3)^{3/2}] + C.$

9. The given line is

$2y + x = 8 \Rightarrow y = 4 - \frac{1}{2}x$ ...(i)

The area between the line $y = 4 - \frac{1}{2}x$, and the x-axis between $x = 3$ and $x = 5$ is represented by region $PLMQ$.



Now, required area $= \int_3^5 y\,dx = \int_3^5\left(4-\frac{1}{2}x\right)dx$

$= \left[4x - \frac{1}{4}x^2\right]_3^5 = \left[4(5) - \frac{25}{4}\right] - \left[12 - \frac{9}{4}\right]$

$= 8 - 4 = 4$ sq. units.

10. $\frac{x\,e^x}{(x+1)^2} = \frac{(x+1)-1}{(x+1)^2}e^x = \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)e^x$

Let $f(x) = \frac{1}{x+1}$, then $f'(x) = -\frac{1}{(x+1)^2}$

Thus, the given integral is of the form $\int (f(x) + f'(x))e^x dx$

$\therefore \int_0^1 \frac{x\,e^x}{(x+1)^2}dx = \int_0^1\left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)e^x\,dx = \left[\frac{1}{x+1}\cdot e^x\right]_0^1$

$= \frac{1}{2}e^1 - \frac{1}{1}e^0 = \frac{1}{2}e - 1.$

11. Given circles are $x^2 + y^2 = 1$ ...(i)

It has centre at (0, 0) and radius = 1

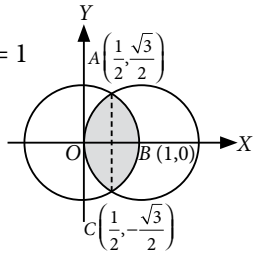
and $(x - 1)^2 + y^2 = 1$. ...(ii)

It has centre at (1, 0) and radius = 1

Subtracting (ii) from (i), we get

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

From (i), when $x = \frac{1}{2}$,



$y^2 = \frac{3}{4} \quad \therefore\ y = \pm\frac{\sqrt{3}}{2}.$

Now, required area = area ($OABCO$) = 2 area ($OABO$)

$= 2\int_0^{1/2}\sqrt{1-(x-1)^2}dx + 2\int_{1/2}^1\sqrt{1-x^2}dx$

$= 2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2}$

$+2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/2}^1$

$= \left[\left(\frac{1}{2}-1\right)\sqrt{1-\frac{1}{4}} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right]$

$+\left[(0+\sin^{-1}1) - \frac{1}{2}\sqrt{1-\frac{1}{4}} - \sin^{-1}\frac{1}{2}\right]$

$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units.

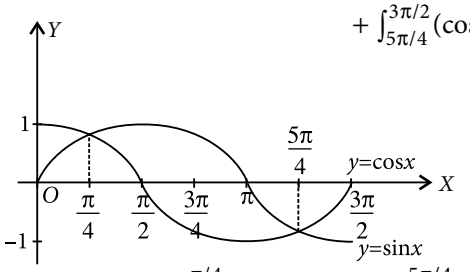
12. We have, $y = \cos x$ ... (i), $0 \le x \le \frac{3\pi}{2}$

and $y = \sin x$ ... (ii), $0 \le x \le \frac{3\pi}{2}$

From (i) and (ii), points of intersections between $\left[0, \frac{3\pi}{2}\right]$ are $\frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

Area $= \int_0^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)dx$

$+ \int_{5\pi/4}^{3\pi/2}(\cos x - \sin x)\,dx$



$= \left|(\sin x + \cos x)\right|_0^{\pi/4} - \left|(\cos x + \sin x)\right|_{\pi/4}^{5\pi/4}$

$+\left|(\sin x + \cos x)\right|_{5\pi/4}^{3\pi/2}$

$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right) - \left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$

$+\left(\left(-1 + \frac{1}{\sqrt{2}}\right) + \left(0 + \frac{1}{\sqrt{2}}\right)\right)$

$= (4\sqrt{2} - 2)$ sq. units.

13. Let $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Then, $dx = (-2\alpha\cos\theta\sin\theta + 2\beta\sin\theta\cos\theta)\,d\theta$

$= 2(\beta - \alpha)\sin\theta\cos\theta\,d\theta$

When $x = \alpha$, we have, $\alpha = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow\ \alpha(1 - \cos^2\theta) = \beta\sin^2\theta \quad \Rightarrow\ \alpha\sin^2\theta = \beta\sin^2\theta$

$\Rightarrow (\alpha - \beta)\sin^2\theta = 0 \quad \therefore \ \theta = 0$

when $x = \beta$, we have, $\beta = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow \beta\cos^2\theta = \alpha\cos^2\theta \Rightarrow (\beta - \alpha)\cos^2\theta = 0$

$\Rightarrow \cos^2\theta = 0 \quad \therefore \ \theta = \pi/2$

Now, $I = \int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta - x)}}$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\alpha\cos^2\theta + \beta\sin^2\theta - \alpha)(\beta - \alpha\cos^2\theta - \beta\sin^2\theta)}} d\theta$$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\beta-\alpha)\sin^2\theta(\beta-\alpha)\cos^2\theta}} d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2} - 0\right] = \pi .$$

14. Let $2x + 3 = A\frac{d}{dx}(x^2 + 4x + 3) + B = A(2x + 4) + B$

On comparing coefficient of x and constant term, we get

$2A = 2$ and $4A + B = 3 \Rightarrow A = 1, B = -1$

Let $I = \int (2x+3)\sqrt{x^2+4x+3}dx$

$= \int (2x+4)\sqrt{x^2+4x+3}dx - \int \sqrt{x^2+4x+3}dx$

$= I_1 - I_2$ (say) ...(i)

For I_1, put $x^2 + 4x + 3 = t \Rightarrow (2x + 4)dx = dt$.

$$\therefore \ I_1 = \int \sqrt{t}dt = \frac{t^{3/2}}{\frac{3}{2}} + C_1 = \frac{2}{3}(x^2+4x+3)^{3/2} + C_1 \quad ...(ii)$$

and $I_2 = \int \sqrt{x^2+4x+3}\ dx = \int \sqrt{(x+2)^2 - 1^2}dx$

$= \int \sqrt{u^2 - 1^2}du$ (Put $x + 2 = u \Rightarrow dx = du$)

$$= \frac{u}{2}\sqrt{u^2-1^2} - \frac{1^2}{2}\log|u + \sqrt{u^2-1^2}| + C_2$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2}\log|x+2+\sqrt{x^2+4x+3}| + C_2$$

...(iii)

From (i), (ii) and (iii), we get

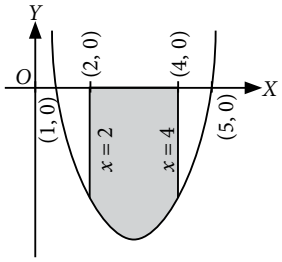
$$I = \frac{2}{3}(x^2+4x+3)^{3/2}$$

$$-\frac{1}{2}(x+2)\sqrt{x^2+4x+3} + \frac{1}{2}\log|x+2+\sqrt{x^2+4x+3}| + C.$$

15. The given curve is $y = x^2 - 6x + 5$.

$\Rightarrow y - (-4) = (x-3)^2$,

which represents an upward parabola with vertex at (3, –4).



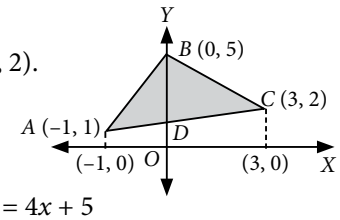
$\therefore$ Required area

$$= \left|\int_2^4 y\,dx\right| = \left|\int_2^4 (x^2 - 6x + 5)dx\right|$$

$$= \left|\left[\frac{x^3}{3} - 6\cdot\frac{x^2}{2} + 5x\right]_2^4\right| = \left|\left(\frac{64}{3} - 48 + 20\right) - \left(\frac{8}{3} - 12 + 10\right)\right|$$

$= 22/3$ sq. units.

16. Given points are $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.



Equation of AB is

$y - 1 = \frac{5-1}{0+1}(x + 1)$

or $y - 1 = 4(x + 1)$ or $y_1 = 4x + 5$

Equation of BC is $y - 5 = \frac{5-2}{0-3}(x - 0)$

or $y_2 = 5 - x$

Equation of AC is $y - 1 = \frac{2-1}{3+1}(x + 1)$

or $y_3 = \frac{x+5}{4}$

Now, area of ΔABC = area $ABDA$ + area $DBCD$

$$= \int_{-1}^0 (y_1 - y_3)dx + \int_0^3 (y_2 - y_3)dx$$

$$= \int_{-1}^0 \left[4x + 5 - \frac{x+5}{4}\right]dx + \int_0^3 \left[5 - x - \frac{x+5}{4}\right]dx$$

$$= [2x^2 + 5x]_{-1}^0 - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^0 + \left[5x - \frac{x^2}{2}\right]_0^3 - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_0^3$$

$$= [0 - (2 - 5)] - \frac{1}{4}\left[0 - \left(\frac{1}{2} - 5\right)\right] + \left[\left(15 - \frac{9}{2}\right) - 0\right]$$

$$-\frac{1}{4}\left[\left(\frac{9}{2} + 15\right) - 0\right]$$

$$= 3 - \frac{9}{8} + \frac{21}{2} - \frac{39}{8} = 3 + \frac{21}{2} - 6 = \frac{15}{2} \text{ sq. units.}$$

17. Let $I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{1 + \cos(\pi/2 - x) + \sin(\pi/2 - x)}dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{1 + \sin x + \cos x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos x + \sin x}{1 + \sin x + \cos x}dx$$

$$= \int_0^{\pi/2} \left\{1 - \frac{1}{1 + \sin x + \cos x}\right\}dx$$

$$= \int_0^{\pi/2} 1\cdot dx - \int_0^{\pi/2} \frac{1}{1 + \sin x + \cos x}dx$$

$$= [x]_0^{\pi/2} - \int_0^{\pi/2} \frac{1 + \tan^2 x/2}{1 + \tan^2 x/2 + 2\tan x/2 + 1 - \tan^2 x/2}dx$$

$$= \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

$$= \frac{\pi}{2} - \int_0^1 \frac{2dz}{2+2z} \text{ (where } z = \tan\frac{x}{2}) = \frac{\pi}{2} - [\log(1+z)]_0^1$$

$$\therefore \quad 2I = \frac{\pi}{2} - \log 2 \quad \Rightarrow \quad I = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

18. (i) Let $I = \int \sec^{-1}\sqrt{x}\,dx = \int \sec^{-1}\sqrt{x}\,.1\,dx$

Integrating by parts, taking $\sec^{-1}\sqrt{x}$ as the first function, we get

$$I = \sec^{-1}\sqrt{x}\cdot x - \int \frac{1}{\sqrt{x}\sqrt{(\sqrt{x})^2-1}}.\frac{1}{2}.x^{-1/2}.x\,dx$$

$$= x\sec^{-1}\sqrt{x} - \frac{1}{2}\int \frac{dx}{\sqrt{x-1}} = x\sec^{-1}\sqrt{x} - \frac{1}{2}\int (x-1)^{-1/2}\,dx$$

$$= x\sec^{-1}\sqrt{x} - \frac{1}{2}\cdot\frac{(x-1)^{1/2}}{1/2} + C = x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C.$$

(ii) Let $I = \int (1+x^2)\sin 3x\,dx$

$$\therefore \quad I = (1+x^2).\left(-\frac{\cos 3x}{3}\right) - \int 2x.\left(-\frac{\cos 3x}{3}\right)dx$$

(Integrating by parts, taking $(1+x^2)$ as the first function)

$$= -\frac{1}{3}(1+x^2)\cos 3x + \frac{2}{3}\int x\cos 3x\,dx$$

(Again integrating by parts)

$$= -\frac{1}{3}(1+x^2)\cos 3x + \frac{2}{3}\left(x\cdot\frac{\sin 3x}{3} - \int 1.\frac{\sin 3x}{3}dx\right)$$

$$= -\frac{1}{3}(1+x^2)\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\left(-\frac{\cos 3x}{3}\right) + C$$

$$= -\frac{1}{27}(7+9x^2)\cos 3x + \frac{2}{9}x\sin 3x + C.$$

19. We know, $\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$ ...(i)

Where, $nh = b - a$ and $n \to \infty$

Here, $f(x) = x^2 - 2x + 2$, $a = 0$ and $b = 3$

$\therefore \quad f(a+rh) = f(0+rh) = f(rh) = r^2h^2 - 2rh + 2$

Also, $nh = b - a = 3 - 0 = 3$

Now from (i), $\int_0^3 (x^2-2x+2)dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h(r^2h^2 - 2rh + 2)$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1}(h^3r^2 - 2h^2r + 2h)$$

$$= \lim_{h\to 0}\left(h^3\sum_{r=0}^{n-1}r^2 - 2h^2\sum_{r=0}^{n-1}r + 2h\sum_{r=0}^{n-1}1\right)$$

$$= \lim_{h\to 0}[h^3(1^2 + 2^2 + \ldots + (n-1)^2) - 2h^2(1 + 2 + 3 + \ldots + (n-1)) + 2hn]$$

$$= \lim_{h\to 0}\left[h^3\frac{n(n-1)(2n-1)}{6} - 2h^2\frac{n(n-1)}{2} + 2nh\right]$$

$$= \lim_{h\to 0}\left[\frac{nh(nh-h)(2nh-h)}{6} - nh(nh-h) + 2nh\right]$$

$$= \lim_{h\to 0}\left[\frac{3(3-h)(2\times 3-h)}{6} - 3(3-h) + 2\times 3\right] \quad (\because nh = 3)$$

$$= \frac{3\times 3\times 6}{6} - (3\times 3) + 2\times 3 = 6$$

20. Let $I = \int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx = \int \frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}dx$

$$= \int \frac{x}{(x^2+1)(x+1)}dx + 2\int \frac{dx}{(1+x^2)^2} = I_1 + I_2 \qquad \ldots(i)$$

For I_1, let $\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$

$\therefore \quad x = A(1+x^2) + (Bx+C)(1+x)$

Putting $x = -1$, we get $A = -\frac{1}{2}$

Putting $x = 0$, we get, $0 = A + C \Rightarrow C = -A = \frac{1}{2}$

Putting $x = 1$, we get $1 = 2A + 2B + 2C \Rightarrow B = \frac{1}{2}$

$$\therefore \quad I_1 = \int\left(-\frac{1}{2(1+x)} + \frac{\frac{1}{2}x + \frac{1}{2}}{1+x^2}\right)dx$$

$$= -\frac{1}{2}\log|1+x| + \frac{1}{2}\int\frac{x}{1+x^2}dx + \frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= -\frac{1}{2}\log|1+x| + \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1}x + C_1 \qquad \ldots(ii)$$

For $I_2 = \int\frac{2dx}{(1+x^2)^2}$, put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

Then, $I_2 = \int\frac{2\sec^2\theta}{(1+\tan^2\theta)^2}d\theta$

$$= \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\,d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right] = [\theta + \sin\theta\cos\theta]$$

$$= \left[\tan^{-1}x + \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right] = \tan^{-1}x + \frac{x}{(1+x^2)} + C_2$$

...(iii)

Now from (i), (ii) and (iii), we get

$$I = \frac{3}{2}\tan^{-1}x - \frac{1}{2}\log|1+x| + \frac{1}{4}\log(1+x^2) + \frac{x}{1+x^2} + C.$$

MPP-5 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Application of Derivatives

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The function $f(x) = x\sqrt{(ax - x^2)}$, $a > 0$
(a) increases on the interval $(0, 3a/4)$
(b) decreases on the interval $(0, 3a/4)$
(c) increases on the interval $(3a/4, a)$
(d) none of these

2. If the function $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 2$ has a positive point of maximum, then $a > 0$ lies in
(a) $(0, \infty)$ (b) $(1, 7)$ (c) $(3, \infty)$ (d) $\left(3, \frac{29}{7}\right]$

3. x and y are the sides of two squares such that $y = x - x^2$. The rate of change of area of the second square with respect to that of the first square is
(a) $2x^2 + 3x + 1$ (b) $3x^2 + 2x - 1$
(c) $2x^2 - 3x + 1$ (d) $3x^2 + 2x + 1$

4. Let $f(x)$ be a quadratic expression which is positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x,
(a) $g(x) < 0$ (b) $g(x) > 0$
(c) $g(x) = 0$ (d) none of these

5. The value of parameter 'a' so that the line $(3 - a)x + ay + (a^2 - 1) = 0$ is normal to the curve $xy = 1$, may lie in the interval
(a) $(-\infty, 0) \cup (3, \infty)$ (b) $(1, 3)$
(c) $(-3, 3)$ (d) none of these

6. The minimum & maximum value of $f(x) = \sin(\cos x) + \cos(\sin x)\ \forall\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ are respectively
(a) cos 1 and 1 + sin 1
(b) sin 1 and 1 + cos 1
(c) cos 1 and $\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)$
(d) none of these

One or More Than One Option(s) Correct Type

7. Let $f: R \to (0, \infty)$ and $g: R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0, f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then
(a) f has a local minimum at $x = 2$
(b) f has a local maximum at $x = 2$
(c) $f''(2) > f(2)$
(d) $f(x) - f''(x) = 0$ for at least one $x \in R$

8. Let $f(x)$ be a differentiable function and $f(a) = f(b) = 0$ $(a < b)$, then which of the following can be true in the interval (a,b)?
(a) $f(x) + f'(x) = 0$ has at least one root.
(b) $f(x) - f'(x) = 0$ has at least one root.
(c) $f(x) \cdot f'(x) = 0$ has at least one root.
(d) none of these

9. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is
(a) increasing on $[0, \infty)$
(b) decreasing on $[0, \infty)$
(c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$
(d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

10. If the tangent to the curve $2y^3 = a x^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, (where $\alpha^2 + \beta^2 = 61$), then the value of a is
(a) -30 (b) 10 (c) 20 (d) 30

11. Let $f''(x) > 0\ \forall\ x \in R$ and $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$. Then,
(a) g attains maximum at $x = \pm 1$
(b) g attains minimum at $x = \pm 1$
(c) g attains maximum at $x = 0$
(d) g attains minimum at $x = \pm 2$

12. The interval to which b may belong so that the function $f(x) = \left(1 - \frac{\sqrt{(21-4b-b^2)}}{b+1}\right)x^3 + 5x + \sqrt{6}$ is increasing at every point of its domain is

(a) $[-7, -1)$ (b) $[-6, -2]$

(c) $[2, 2.5]$ (d) $[2, 3]$

13. Which of the following statements is true for the curve $x = 2a \sin t + a \sin t \cos^2 t$ and $y = -a \cos^3 t$?

(a) Normal is inclined at an angle $\frac{\pi}{2} + t$ with x-axis.

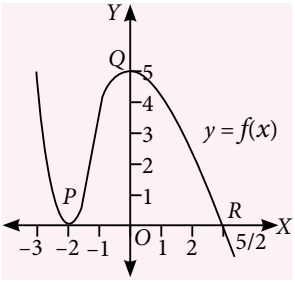
(b) Normal is inclined at an angle t with x-axis.

(c) Portion of normal contained between the co-ordinate axes is equal to $2a$.

(d) Portion of normal contained between the co-ordinate axes is equal to $4a$.

Comprehension Type

A polynomial of degree three is shown in the given figure and at Q its gradient is 3. Then,



14. The polynomial $f(x)$ is given by

(a) $-\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$

(b) $-\frac{1}{2}x^3 + 3x + 5$

(c) $-\frac{1}{2}x^3 - \frac{3}{4}x^2 + 5$ (d) $-\frac{1}{2}x^3 + \frac{3}{4}x^2 - 3x + 5$

15. The equation of normal at R is

(a) $6x - 81y - 15 = 0$

(b) $8x - 81y - 20 = 0$

(c) $2x - 81y - 5 = 0$

(d) $4x - 81y - 10 = 0$

Matrix Match Type

16. Match the following:

Column-I		Column-II	
P.	An open box with square base of side x and height y has a given surface area. Its volume is maximum, if $\frac{x}{y} =$	1.	24
Q.	A cylindrical can with lid having base radius x and height y has a given surface area. Its volume is maximum, if $\frac{6y}{x} =$	2.	3
R.	Curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may touch, if $\frac{3c}{a+b}$ is equal to	3.	2
S.	Circular plate is expanded by heat from radius 6 cm to 6.06 cm. If approximate increase in area is $\frac{3k\pi}{100}$, then k equals	4.	12

	P	Q	R	S
(a)	1	3	4	2
(b)	2	3	1	4
(c)	4	2	3	1
(d)	3	4	2	1

Integer Answer Type

17. The maximum possible integral value of $\frac{\beta - \alpha}{\tan^{-1}\beta - \tan^{-1}\alpha}$, where $0 < \alpha < \beta < \sqrt{3}$, is

18. If $y = a \ln x + bx^2 + x$ has its extreme values at $x = -1$, 2 then 2 $(a + b)$ is equal to

19. If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)

20. The chord of the curve $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point $x = 2$ and is bisected by that point. Then the value of a is ◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MOCK TEST PAPER 2019

JEE MAIN

Time : 1 hr 15 min. **Series - 4**

The entire syllabus of Mathematics of JEE MAIN is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

	Topic	Syllabus In Detail
UNIT NO. 4	Mathematical Induction	Principle of mathematical induction and its simple application
	Binomial Theorem & it's Simple Applications	Binomial Theorem of positive integral index, general term and middle term, properties of Binomial coefficients and simple applications
	Sequence and series	Definition of series and sequence. Arithmetic, geometric and harmonic progressions. Arithmetic, geometric & harmonic means between two given numbers, relation between A.M,G.M and H.M. Sum up to n terms of special series: $\sum n, \sum n^2, \sum n^3$. Arithmetic- Geometric progression.
	Co-ordinate geometry-2D	Conic Sections: Sections of cones, equation conic sections(parabola, ellipse and hyperbola) in standard forms, condition for $y = mx + c$ to be a tangent and point(s) of tangency

1. If $49^n + 16n + \lambda$ is divisible by 64 for all $n \in N$, then the least negative integral value of λ is

(a) -2 (b) -1 (c) -3 (d) -4

2. $\frac{1}{1^3}\cdot\frac{2}{2} + \frac{2}{1^3+2^3}\cdot\frac{3}{2} + \frac{3}{1^3+2^3+3^3}\cdot\frac{4}{2} + \ldots\ldots$upto n terms $=$

(a) $\frac{n-1}{n}$ (b) $\frac{n}{n+1}$ (c) $\frac{n+1}{n+2}$ (d) $\frac{n+1}{n}$

3. The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by

(a) $3 \ \forall \ n \in N$ (b) $4 \ \forall \ n \in N$

(c) $5 \ \forall \ n \in N$ (d) can't be determined

4. Let $P(n) : n^2 + n + 1$ is an even integer. If $P(k)$ is assumed true $\Rightarrow P(k+1)$ is true. Therefore, $P(n)$ is

(a) true for $n > 1$ (b) true for all $n \in N$

(c) true for $n > 2$ (d) none of these

5. For all $n \in N$, $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta$ equals to

(a) $\frac{\sin 2^n\theta}{2^n \sin\theta}$ (b) $\frac{\sin 2^n\theta}{\sin\theta}$

(c) $\frac{\cos 2^n\theta}{2^n \cos 2\theta}$ (d) $\frac{\cos 2^n\theta}{2^n \sin\theta}$

6. If the coefficients of x and x^2 in the expansion of $(1 + x)^m (1 - x)^n$, $m, n \in N$ are 3 and -6 respectively, then value of m is

(a) 6 (b) 9

(c) 12 (d) 24

7. The coefficient of x^{100} in the expansion of $\sum_{r=0}^{200} (1+x)^r$ is

(a) ${}^{200}C_{100}$ (b) ${}^{201}C_{102}$

(c) ${}^{200}C_{101}$ (d) ${}^{201}C_{101}$

8. The coefficient of $x^r (0 \le r \le n-1)$ in the expansion of $(x + 2)^{n-1} + (x + 2)^{n-2}(x + 1) + (x + 2)^{n-3}(x + 1)^2 + \ldots + (x + 1)^{n-1}$, is

(a) ${}^nC_r 2^{n-r}$ (b) ${}^nC_r (2^{n-r} - 1)$

(c) ${}^nC_r (2^{n-r} + 1)$ (d) ${}^nC_r (2^n - 1)$

By : Sankar Ghosh, S.G.M.C, Mob. 09831244397.

9. The number of irrational terms in the expansion of $(\sqrt[8]{5}+\sqrt[6]{2})^{100}$, is

(a) 97 (b) 98 (c) 96 (d) 99

10. The greatest value of the term independent of x in the expansion of $(x\sin\alpha + x^{-1}\cos\alpha)^{10}$, $\alpha \in R$, is

(a) 2^5 (b) $\dfrac{10!}{(5!)^2}$

(c) $\dfrac{10!}{2^5\times(5!)^2}$ (d) none of these.

11. The coefficients of the middle terms in the expansion of $(1+\alpha x)^4$ and $(1-\alpha x)^4$ is same, if α equals

(a) $\dfrac{2}{5}$ (b) $\dfrac{10}{3}$ (c) $-\dfrac{3}{10}$ (d) $-\dfrac{5}{3}$

12. The larger of $99^{50}+100^{50}$ and 101^{50}, is

(a) $99^{50}+100^{50}$ (b) both are equal

(c) 101^{50} (d) none of these

13. If $\alpha \neq 1$ is n^{th} root of unity and $n \in N$ such that first three terms in the expansion of $(\alpha + x)^n$ are $1, \bar{\alpha}$ and $\dfrac{(n-1)}{2n}\bar{\alpha}^2$, then the value x is

(a) $1/n$ (b) $2/n$ (c) $1/2$ (d) $1/4$

14. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then value of n equals

(a) 5 (b) 7 (c) 6 (d) 8

15. The number of real negative terms in the binomial expansion of $(1+ix)^{4n-2}$, $n \in N$ is

(a) n (b) $n+1$

(c) $n-1$ (d) $2n$

16. Let a_n be the n^{th} term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, $\alpha \neq \beta$, then the common ratio is

(a) $\dfrac{\alpha}{\beta}$ (b) $\dfrac{\beta}{\alpha}$ (c) $\sqrt{\dfrac{\alpha}{\beta}}$ (d) $\sqrt{\dfrac{\beta}{\alpha}}$

17. For $0<\theta<\dfrac{\pi}{2}$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\theta$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$, $z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\phi$, then

(a) $xy = zx + zy + z$ (b) $xy = zx + zy - z$

(c) $xy + zy + zx = z$ (d) none of these.

18. The sum to infinity of the series $1+\dfrac{2}{3}+\dfrac{6}{3^2}+\dfrac{10}{3^3}+\dfrac{14}{3^4}+\ldots$ is

(a) 2 (b) 3 (c) 4 (d) 6

19. If $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$, and the sum to infinite terms of the series $\cos x+\dfrac{2}{3}\cos x\sin^2 x+\dfrac{4}{9}\cos x\sin^4 x+\ldots$ is finite, then

(a) $x\in\left(-\dfrac{\pi}{3},\dfrac{\pi}{3}\right)$ (b) $x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

(c) $x\in\left(-\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$ (d) none of these

20. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$ the length of side of S_n equals the length of diagonal of S_{n+1}. If the length of side of S_1 is 10 cm, then for each of the following values of n is the area of S_n less than 1 sq.cm?

(a) 7 (b) 8 (c) 5 (d) 6

21. In a sequence of $(4n+1)$ terms, the first $(2n+1)$ terms are in A.P. whose common difference is 2, and the last $(2n+1)$ terms are in G.P. whose common ratio is 0.5. If the middle terms of the A.P. and G.P. are equal, then the middle of the sequence is

(a) $\dfrac{n\cdot 2^{n+1}}{2^n-1}$ (b) $\dfrac{n\cdot 2^{n+1}}{2^{2n}-1}$

(c) $n\cdot 2^n$ (d) none of these

22. Let there be a G.P. whose first term is a and the common ratio is r. If A and H are the arithmetic mean and the harmonic mean respectively for the first n terms of the G.P. Then, AH is equal to

(a) $a^2 r^{n-1}$ (b) ar^n

(c) a^2r^n (d) none of these

23. If $\dfrac{1}{\sqrt{x-1}}+\dfrac{1}{\sqrt{y-1}}+\dfrac{1}{\sqrt{z-1}}>0$ and x, y, z are in G.P., then $(\log x^2)^{-1}, (\log xz)^{-1}, (\log z^2)^{-1}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

24. If three successive terms of a G.P. with common ratio $r\,(r>1)$ form the sides of a ΔABC and $[r]$ denotes greatest integer function, then $[r]+[-r]$ is

(a) 0 (b) 1

(c) -1 (d) none of these

25. If a, x, b are in A.P., a, y, b are in G.P. and a, z, b are in H.P. such that $x = 9z$ and $a > 0, b > 0$, then
(a) $|y| = 3z$ and $x = 3|y|$
(b) $y = 2|z|$ and $|x| = 3y$
(c) $2y = x + z$
(d) none of these

26. If (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then its equation is
(a) $x^2 + 8y = 32$ (b) $y^2 + 8x = 32$
(c) $x^2 - 8y = 32$ (d) $y^2 - 8x = 32$

27. If $(a^2, a - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, –4), then the set of all possible values of a, is
(a) $(-2, \sqrt{2})$ (b) $(-3, 2)$
(c) $(-2, 2\sqrt{2})$ (d) $(-2+2\sqrt{2}, \sqrt{2})$

28. The area (in sq.units) of the quadrilateral formed by the tangents at the end-points of latusrectum of $\frac{x^2}{9}+\frac{y^2}{5}=1$ is equal to
(a) $\frac{27}{4}$ (b) 9
(c) $\frac{27}{2}$ (d) 27

29. If e_1 is the eccentricity of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ and e_2 be the eccentricity of the hyperbola passing through the focus of the ellipse such that $e_1e_2 = 1$, then equation of the hyperbola is
(a) $\frac{x^2}{9}-\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}-\frac{y^2}{9}=-1$
(c) $\frac{x^2}{9}-\frac{y^2}{25}=1$ (d) none of these

30. PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of the hyperbola satisfies
(a) $1<e<\frac{2}{\sqrt{3}}$ (b) $e=\frac{2}{\sqrt{3}}$
(c) $e=\frac{\sqrt{3}}{2}$ (d) $e>\frac{2}{\sqrt{3}}$

SOLUTIONS

1. (b) : For $n = 1$, we have
$49^n + 16n + \lambda = 49 + 16 + \lambda = 65 + \lambda$
$= 64 + (\lambda + 1)$
This is divisible by 64 if $\lambda = -1$.
For $n = 2$, we have
$49^n + 16n + \lambda = 49^2 + 16 \times 2 + \lambda = 2433 + \lambda$
$= 64 \times 38 + (\lambda + 1)$
This is divisible by 64 if $\lambda = -1$.
Hence, $\lambda = -1$

2. (b) : We have, $T_n=\frac{n(n+1)}{4\Sigma n^3}=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$\therefore\ T_1=\frac{1}{1}-\frac{1}{2}, T_2=\frac{1}{2}-\frac{1}{3}, \ldots\ldots, T_n=\frac{1}{n}-\frac{1}{n+1}$

Adding above equations, we get

$S_n=T_1+T_2+\ldots\ldots+T_n=1-\frac{1}{n+1}=\frac{n}{n+1}$.

3. (a) : Let $P(n)$ be the statement given by
$P(n) : S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3
We have, $P(1) : S_1 = 1^3 + 3(1)^2 + 5(1) + 3$
$= 12$, which is divisible by 3
$\therefore$ $P(1)$ is true.
Let $P(k)$ be true. Then,
$P(k) : S_k = k^3 + 3k^2 + 5k + 3$ is divisible by 3
$\Rightarrow$ $S_k = k^3 + 3k^2 + 5k + 3 = 3\lambda$, for some $\lambda \in N$ ...(i)
We now wish to show that $P(k + 1)$ is true, *i.e.*
$(k + 1)^3 + 3(k + 1)^2 + 5(k + 1) + 3$ is divisible by 3
Now, $(k + 1)^3 + 3(k + 1)^2 + 5(k + 1) + 3$
$= (k^3 + 3k^2 + 5k + 3) + 3k^2 + 9k + 9$
$= 3\lambda + 3(k^2 + 3k + 3)$ [Using (i)]
$= 3[\lambda + k^2 + 3k + 3]$, which is divisible by 3
Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.
Hence, by the principle of mathematical induction, the statement is true for all $n \in N$.

4. (d) : Given, $P(n) : n^2 + n + 1$ is an even integer.
For $n = 1$, $P(1) = 3$, which is not an even integer.
$\therefore$ $P(1)$ is not true.
Also, $n^2 + n + 1 = n(n + 1) + 1$ is always an odd integer.
(principle of induction is not applicable)

5. (a) : Let $P(n) = \cos\theta \cos 2\theta \ldots\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$

$P(1)=\cos\theta=\frac{\sin 2\theta}{2\sin\theta}=\cos\theta$, true

$P(n+1)=(\cos\theta\cos 2\theta\cos 2^2\theta\ldots\ldots\cos 2^{n-1}\theta)\times\cos 2^n\theta$

$=\frac{\sin 2^n\theta}{2^n\sin\theta}\times\cos 2^n\theta=\frac{\sin 2\cdot 2^n\theta}{2\cdot 2^n\sin\theta}=\frac{\sin 2^{n+1}\theta}{2^{n+1}\sin\theta}$

$\therefore$ By P.M.I., $P(n)$ is true for all $n \in N$.

6. (c) : We have, $(1+x)^m(1-x)^n$

$=({}^mC_0+{}^mC_1x+{}^mC_2x^2+...+{}^mC_mx^m)\times({}^nC_0-{}^nC_1x+{}^nC_2x^2+...+(-1)^n{}^nC_nx^n)$

$=({}^mC_0\,{}^nC_0)-({}^mC_0\,{}^nC_1-{}^nC_0\,{}^mC_1)x+({}^mC_0\,{}^nC_2+{}^nC_0\,{}^mC_2-{}^mC_1\,{}^nC_1)x^2+...$

Now, according to problem, we have

$-({}^mC_0\,{}^nC_1-{}^nC_0\,{}^mC_1)=3$

Also, $({}^mC_0\,{}^nC_2+{}^nC_0\,{}^mC_2-{}^mC_1\,{}^nC_1)=-6$

$\Rightarrow$ $m-n=3$ and $n(n-1)+m(m-1)-2mn=-12$

$\Rightarrow$ $m-n=3$ and $(m-n)^2-(m+n)=-12$

$\Rightarrow$ $m-n=3$ ...(i)

and $m+n=21$ ...(ii)

Solving (i) and (ii), we get $m=12$, $n=9$.

7. (d) : $\sum_{r=0}^{200}(1+x)^r=1+(1+x)+(1+x)^2+...+(1+x)^{200}$

$\Rightarrow \sum_{r=0}^{200}(1+x)^r=\left\{\frac{(1+x)^{201}-1}{(1+x)-1}\right\}=\frac{1}{x}\{(1+x)^{201}-1\}$

$\therefore$ Coefficient of x^{100} in $\sum_{r=0}^{200}(1+x)^r$

= Coefficient of x^{101} in $(1+x)^{201}={}^{201}C_{101}$.

8. (b) : $(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2+...+(x+1)^{n-1}$

$=(x+2)^{n-1}\left\{\frac{1-\left(\frac{x+1}{x+2}\right)^n}{1-\frac{x+1}{x+2}}\right\}=(x+2)^n-(x+1)^n$

$\therefore$ Coefficient of x^r in the given expansion

= coeff. of x^r in $(x+2)^n$ − coeff. of x^r in $(x+1)^n$

$={}^nC_r\,2^{n-r}-{}^nC_r=(2^{n-r}-1)\,{}^nC_r$

9. (a) : $\left(\sqrt[8]{5}+\sqrt[6]{2}\right)^{100}=\sum_{r=0}^{100}{}^{100}C_r\left(\sqrt[8]{5}\right)^{100-r}\left(\sqrt[6]{2}\right)^r$

$=\sum_{r=0}^{100}{}^{100}C_r\,5^{\frac{100-r}{8}}\,2^{\frac{r}{6}}=\sum_{r=0}^{100}T_{r+1}$

Where $T_{r+1}={}^{100}C_r\,5^{\frac{100-r}{8}}\,2^{\frac{r}{6}}$

Clearly T_{r+1}, will be an integer if $\frac{100-r}{8}$ and $\frac{r}{6}$ are integers.

This is possible when

$100-r$ is a multiple of 8 and r is a multiple of 6

$\Rightarrow$ $100-r=0, 8, 16, ...,96$ and $r=0, 6, 12, ..., 96$

$\Rightarrow$ $r=4, 12, 20, ..., 100$ and $r=0, 6, 12, ..., 96$

$\Rightarrow$ $r=12, 36, 60, 84$

Hence, there are 4 rational terms and $101-4=97$ irrational terms.

10. (c) : Let $(r+1)^{\text{th}}$ term be independent of x.

Now, $T_{r+1}={}^{10}C_r(x\sin\alpha)^{10-r}\left(\frac{\cos\alpha}{x}\right)^r$

$={}^{10}C_r\,x^{10-2r}(\sin\alpha)^{10-r}(\cos\alpha)^r$

This is independent of x when $r=5$.

Therefore, term independent of x is,

$T_6={}^{10}C_5(\sin\alpha)^5(\cos\alpha)^5={}^{10}C_5\,2^{-5}(\sin2\alpha)^5$

Clearly, T_6 will be greatest when $2\alpha=\frac{\pi}{2}$ and its greatest value is ${}^{10}C_5\,2^{-5}=\frac{10!}{2^5(5!)^2}$.

11. (c) : The middle term in the expansion of

$(1+\alpha x)^4={}^4C_2(\alpha x)^2=6\alpha^2x^2$

The middle term in the expansion of

$(1-\alpha x)^6={}^6C_3(-\alpha x)^3=-20\alpha^3x^3$

According to question, we have,

$6\alpha^2=-20\alpha^3\Rightarrow\alpha=0,-\frac{3}{10}$

12. (c) : $99^{50}+100^{50}-101^{50}=(100-1)^{50}-(100+1)^{50}+100^{50}$

$=-\{(100+1)^{50}-(100-1)^{50}\}+100^{50}$

$=-2({}^{50}C_1\times100+{}^{50}C_3\times100^3+...+{}^{50}C_{49}\times100^{49})+100^{50}<0$

$(\because -2\{{}^{50}C_1\times100+{}^{50}C_3\times100^3+...+{}^{50}C_{49}\times100^{49}\}<0)$

$\therefore$ $99^{50}+100^{50}-101^{50}<0\Rightarrow99^{50}+100^{50}<101^{50}$

13. (a) : Let $\alpha=e^{i\frac{2k\pi}{n}}$, where $k=1, 2, 3, ..., n-1$.

Then, $\alpha^{n-1}=e^{i\frac{2k\pi(n-1)}{n}}=e^{\frac{-i2k\pi}{n}}=\bar{\alpha}$ and $\alpha^{n-2}=(\bar{\alpha})^2$.

Now, $(\alpha+x)^n={}^nC_0\alpha^nx^0+{}^nC_1\alpha^{n-1}x+{}^nC_2\alpha^{n-2}x^2+....$

$\Rightarrow (\alpha+x)^n=1+n\bar{\alpha}x+\frac{n(n-1)}{2}(\bar{\alpha})^2x^2+....$ $[\because \alpha^n=1]$

MPP-5 CLASS XII ANSWER KEY

1.	(a)	**2.**	(d)	**3.**	(c)	**4.**	(b)	**5.**	(a)
6.	(a)	**7.**	(a,d)	**8.**	(a,b,c)	**9.**	(b)	**10.**	(a,d)
11.	(c,d)	**12.**	(a,b,c,d)			**13.**	(a, c)	**14.**	(a)
15.	(b)	**16.**	(d)	**17.**	(3)	**18.**	(3)	**19.**	(5)
20.	(1)								

$\therefore \quad nx = 1$ ...(i) and $\frac{n(n-1)}{2}x^2 = \frac{n-1}{2n}$ ...(ii)

From (i) and (ii), we get $x = \frac{1}{n}$.

14. (c) : Let the three consecutive terms in the expansion be r^{th}, $(r+1)^{th}$, $(r+2)^{th}$.

Given that, ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5:10:14$

$\Rightarrow \quad \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5}$ and $\frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$

$\Rightarrow \quad \frac{n+5-r+1}{r} = 2$ and $\frac{n+5-r}{r+1} = \frac{7}{5}$

$\Rightarrow \quad n - 3r + 6 = 0$ ...(i)

and $5n - 12r + 18 = 0$ ...(ii)

Solving (i) and (ii), we get, $n = 6$, $r = 4$.

15. (a) : Let T_{r+1} be the $(r+1)^{th}$ term in the expansion $(1 + ix)^{4n-2}$.

$\therefore \quad T_{r+1} = {}^{4n-2}C_r (ix)^r = {}^{4n-2}C_r x^r i^r$

Clearly, T_{r+1} is real negative, if $r = 2, 6, 10, 14, ...$

Here the values of r are in A.P. with first term 2 and common difference 4.

Its k^{th} term is $2 + (k-1)4 = 4k - 2$.

As r lies between 2 and $4n - 2$.

$\therefore \quad 2 \le 4k - 2 \le 4n - 2 \quad \Rightarrow \quad 1 \le k \le n$

Hence, there are n real negative terms.

16. (a) : Let a be the first term and r be the common ratio of the G.P.

Now, $\alpha = \sum_{n=1}^{100} a_{2n} = a_2 + a_4 + a_6 + ... + a_{200}$

$= ar + ar^3 + ar^5 + ... + ar^{199}$

$\Rightarrow \quad \alpha = ar(1 + r^2 + r^4 + ... + r^{198})$ ...(i)

and, $\beta = \sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + ... + a_{199}$

$= a + ar^2 + ar^4 + ... + ar^{198}$ ...(ii)

$\Rightarrow \quad \beta = a(1 + r^2 + r^4 + ... + r^{198})$

From (i) and (ii), we get, $\frac{\alpha}{\beta} = r$.

17. (b) : Given, $x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + ...\infty$

$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + ...\infty$

and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi = 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + ...\infty$

$\therefore \quad x = \frac{1}{1-\cos^2\theta}, y = \frac{1}{1-\sin^2\phi}$ and $z = \frac{1}{1-\cos^2\theta\sin^2\phi}$

$\Rightarrow \quad x = \frac{1}{\sin^2\theta}, y = \frac{1}{\cos^2\phi}$ and $z = \frac{1}{1-\cos^2\theta\sin^2\phi}$

$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} = \frac{xy}{xy-(xy-x-y+1)}$

$\Rightarrow \quad z = \frac{xy}{x+y-1} \quad \Rightarrow \quad xy = zx + zy - z$

18. (b) : Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...\infty$

$\Rightarrow \quad S - 1 = \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ... \infty$ ...(i)

$\Rightarrow \quad (S-1) \times \frac{1}{3} = \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \frac{14}{3^5} + ...\infty$ ...(ii)

Subtracting (ii) from (i), we get

$\frac{2}{3}(S-1) = \frac{2}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + ...\infty$

$\Rightarrow \quad \frac{2}{3}(S-1) = \frac{2}{3} + \frac{\frac{4}{3^2}}{1-\frac{1}{3}}$

$\Rightarrow \quad \frac{2}{3}(S-1) = \frac{2}{3} + \frac{2}{3} \quad \Rightarrow \quad S = 3.$

19. (b) : The given infinite series is

$\cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + ...$

The above series in an infinite G.P. with common ratio $\frac{2}{3}\sin^2 x$.

The sum of the series will exist when $\left|\frac{2}{3}\sin^2 x\right| < 1$.

Clearly, $\left|\frac{2}{3}\sin^2 x\right| = \frac{2}{3}|\sin x|^2 < \frac{2}{3} < 1$.

Therefore, the sum of the series exists and given by

$S = \frac{\cos x}{1 - \frac{2}{3}\sin^2 x} = \frac{3\cos x}{2 + \cos 2x}$.

$\cos x$ and $2 + \cos 2x$ are finite for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. So, sum of the given series is finite for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

20. (b) : Length of a side of S_n = Length of a diagonal of S_{n+1}

$\Rightarrow$ Length of a side of $S_n = \sqrt{2}$ (Length of a side of S_{n+1})

$\Rightarrow \dfrac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n} = \dfrac{1}{\sqrt{2}}$ for all $n \geq 1$

$\Rightarrow$ Sides of $S_1, S_2, ..., S_n$ forms a G.P. with common ratio $\dfrac{1}{\sqrt{2}}$ and first term 10.

$\therefore$ Length of the side of $S_n = 10\left(\dfrac{1}{\sqrt{2}}\right)^{n-1} = \dfrac{10}{2^{n-1/2}}$

Now, area of $S_n = (\text{side})^2 = \left(\dfrac{10}{2^{n-1/2}}\right)^2 = \dfrac{100}{2^{n-1}}$

For area of $S_n < 1 \Rightarrow \dfrac{100}{2^{n-1}} < 1 \Rightarrow 2^{n-1} > 100$

$\Rightarrow n - 1 \geq 7 \Rightarrow n \geq 8.$

21. (a) : Let the sequence be

$a_1, a_2, a_3, ..., a_{2n}, a_{2n+1}, ..., a_{4n}, a_{4n+1}$.

Then, $a_{2n+1} = a_1 + 4n$ and $a_{4n+1} = a_{2n+1} \times \left(\dfrac{1}{2}\right)^{2n}$

$\Rightarrow a_{2n+1} = a_1 + 4n$ and $a_{4n+1} = \dfrac{a_1 + 4n}{2^{2n}}$

Also given that,

Middle term of first $(2n + 1)$ terms = Middle term of the last $(2n + 1)$ terms

$\Rightarrow a_{n+1} = a_{3n+1} \Rightarrow a_1 + 2n = a_{2n+1}\left(\dfrac{1}{2}\right)^n$

$\Rightarrow a_1 + 2n = (a_1 + 4n) \times \dfrac{1}{2^n} \Rightarrow a_1\left\{1 - \dfrac{1}{2^n}\right\} = \dfrac{4n}{2^n} - 2n$

$\Rightarrow a_1(2^n - 1) = 4n - 2n \times 2^n \Rightarrow a_1 = \dfrac{4n - 2n \times 2^n}{2^n - 1}$

$\therefore$ Middle term of the sequence $= a_{2n+1} = a_1 + 4n$

$= \dfrac{4n - 2n \times 2^n}{2^n - 1} + 4n = \dfrac{2n \times 2^n}{2^n - 1} = \dfrac{n \cdot 2^{n+1}}{2^n - 1}.$

22. (a) : $A = \dfrac{a + ar + ar^2 + ... + ar^{n-1}}{n} = \dfrac{a}{n}\left(\dfrac{1 - r^n}{1 - r}\right)$

and $\dfrac{1}{H} = \dfrac{\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + ... + \frac{1}{ar^{n-1}}}{n}$

$= \dfrac{a^{-1}}{n}\left(\dfrac{r^{-n} - 1}{r^{-1} - 1}\right) = \dfrac{1}{an}\left(\dfrac{1 - r^n}{1 - r}\right)\dfrac{1}{r^{n-1}}$

$\Rightarrow H = anr^{n-1} \times \dfrac{a}{An} \Rightarrow AH = a^2 r^{n-1}.$

23. (c) : Given that, $\dfrac{1}{\sqrt{x-1}} + \dfrac{1}{\sqrt{y-1}} + \dfrac{1}{\sqrt{z-1}} > 0$

$\Rightarrow x > 1, y > 1, z > 1$

Also given that x, y, z are in G.P.

$\therefore$ $\log x, \log y, \log z$ are in A.P.

$\Rightarrow 2\log x, 2\log y, 2\log z$ are in A.P.

$\Rightarrow (\log x^2)^{-1}, (\log y^2)^{-1}, (\log z^2)^{-1}$ are in H.P.

$\Rightarrow (\log x^2)^{-1}, (\log xz)^{-1}, (\log z^2)^{-1}$ are in H.P.

24. (c) : Let the sides of the triangle be a, ar, ar^2. Since $r > 1$, therefore ar^2 is the greatest side.

$\therefore$ $a + ar > ar^2 \Rightarrow ar^2 - a - ar < 0$

$\Rightarrow r^2 - r - 1 < 0 \Rightarrow \dfrac{1 - \sqrt{5}}{2} < r < \dfrac{1 + \sqrt{5}}{2}$

$\Rightarrow 1 < r < \dfrac{1 + \sqrt{5}}{2}$ $[\because r > 1]$

$\Rightarrow [r] = 1$

Also, $-\dfrac{1 + \sqrt{5}}{2} < -r < -1 \Rightarrow [-r] = -2$

Hence, $[r] + [-r] = 1 - 2 = -1$.

25. (a) : Since, a, x, b are in A.P. ; a, y, b are in G.P. and a, z, b are in H.P. , we have x, y, z as A.M., G.M. and H.M. of a and b respectively. Also A.M., G.M. and H.M. are in G.P.

$\therefore$ $y^2 = xz \Rightarrow y^2 = 9z^2$ $[\because x = 9z]$

$\Rightarrow y = 3z \Rightarrow |y| = 3z$ $[\because y > 0 \therefore |y| = y]$

Again, $y^2 = xz$ and $x = 9z$

$\Rightarrow y^2 = \dfrac{x^2}{9} \Rightarrow 9y^2 = x^2$

$\Rightarrow 3y = x \Rightarrow 3|y| = x$ $[\because y > 0 \therefore |y| = y]$

26. (a) : The vertex and focus of the parabola are (0, 4) and (0, 2) respectively. Clearly the axis of the parabola is y-axis and the concavity of the curve is downward.

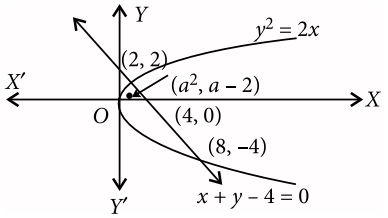
$\because$ The distance between vertex and focus is 2 units.

$\therefore$ Latusrectum = 4 × 2 = 8.

Hence, the equation of the parabola is

$(x - 0)^2 = -8(y - 4) \Rightarrow x^2 + 8y = 32.$

27. (d) : Given that $P(a^2, a - 2)$ lies inside the parabola,



$\therefore$ $(a - 2)^2 - 2 \times a^2 < 0 \Rightarrow a^2 - 4a + 4 - 2a^2 < 0$

$\Rightarrow -a^2 - 4a + 4 < 0 \Rightarrow a^2 + 4a - 4 > 0$

$\Rightarrow (a+2)^2 - (2\sqrt{2})^2 > 0$

$\Rightarrow \quad a+2<-2\sqrt{2} \quad \text{or} \quad a+2>2\sqrt{2}$ ...(i)

Since point $P(a^2, a-2)$ and origin $O(0, 0)$ are on the same side of the chord joining (2, 2) and (8, –4).

$\therefore \quad (0+0-4)(a^2+a-2-4)>0$

$\Rightarrow \quad a^2+a-6<0 \quad \Rightarrow \quad (a+3)(a-2)<0$

$\Rightarrow \quad -3<a<2$ ...(ii)

Also, $-4<a-2<2$ and $0<a^2<8$

$\Rightarrow \quad -2<a<2\sqrt{2}$ ...(iii)

From (i), (ii), and (iii), we get that $a\in(-2+2\sqrt{2}, 2)$.

28. (d) : The equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

$\therefore \quad$ Eccentricity $(e)=\sqrt{1-\frac{5}{9}}=\frac{2}{3}$.

The coordinates of the end-points of latusrectum are

$$L\left(2,\frac{5}{3}\right), L'\left(2,-\frac{5}{3}\right), M\left(-2,\frac{5}{3}\right), M'\left(-2,-\frac{5}{3}\right).$$

The equations of tangents at these points are

$2x+3y-9=0 \Rightarrow y=\frac{-2}{3}x+3$ ...(i)

$2x-3y-9=0 \Rightarrow y=\frac{2}{3}x-3$ ...(ii)

$-2x+3y-9=0 \Rightarrow y=\frac{2}{3}x+3$ ...(iii)

$2x+3y+9=0 \Rightarrow y=\frac{-2}{3}x-3$ ...(iv)

Clearly, the above tangents form a parallelogram whose area is given by

$\because \quad$ Area of $||^{gm}=\left|\frac{(c_1-c_2)(d_1-d_2)}{(m_1-m_2)}\right|$

$\therefore \quad A=\left|\frac{(3+3)\times(-3-3)}{-\frac{2}{3}-\frac{2}{3}}\right|=\frac{6\times6}{4/3}=27$ sq.units.

29. (b) : We have, $e_1=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$.

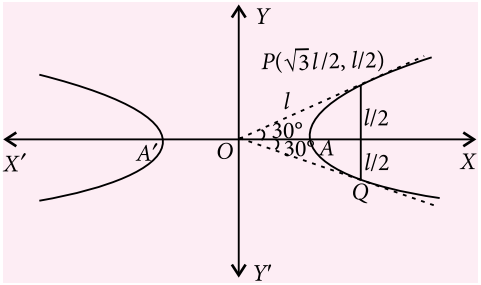
$\because \quad e_1e_2=1 \Rightarrow e_2=\frac{5}{3}$

The coordinates of foci of the ellipse are (0, ±3).

Clearly, hyperbola in option ((b) passes through (0, ±3) and has eccentricity 5/3.

30. (d) : Let the length of each side of the equilateral triangle OPQ be l units. Then, the coordinates of P are $\left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$.

Point $P\left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.



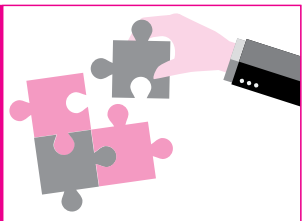
$\therefore \quad \frac{3l^2}{4a^2}-\frac{l^2}{4b^2}=1 \Rightarrow (3b^2-a^2)l^2=4a^2b^2$

$\Rightarrow \quad (3e^2-4)l^2=4a^2(e^2-1) \Rightarrow l=2a\sqrt{\frac{e^2-1}{3e^2-4}}$

Since, l is real and $e > 1$

$\therefore \quad 3e^2-4>0 \Rightarrow e^2>\frac{4}{3} \Rightarrow e>\frac{2}{\sqrt{3}}$.

PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics.

In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is multiple of its numbers. For example, if that value is 3 for a two-box cluster, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

5–	120×		2÷	7+	
	3+				15+
10+		2÷		7+	
			25×		
9+	36×			9+	
			7+		

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

BRAIN @ WORK

Limit, Continuity and Differentiability

LEFT AND RIGHT LIMITS

Let $y = f(x)$ be a given function, and $x = a$ is the point under consideration. Left tendency of $f(x)$ at $x = a$ is called its left limit and right tendency is called its right limit.

$f(a+0) = \lim_{h\to 0} f(a+h)$ and $f(a-0) = \lim_{h\to 0} f(a-h)$

where 'h' is a small positive number.

Thus for the existence of the limit of $f(x)$ at $x = a$, $f(a-0) = f(a+0)$

Reasons for non-existence of the limit

$\lim_{x\to a} f(x)$ will not exist due to any of these three reasons :

(a) $f(x)$ is not defined in the neighbourhood of $x = a$.

(b) Left and right limits of $f(x)$ are not the same.

SOME STANDARD LIMITS

- $\lim_{x\to 0} \frac{\sin ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\sin^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} \frac{\tan^{-1} ax}{x} = a, \ \forall \ a \in R$
- $\lim_{x\to 0} (1+ax)^{1/x} = e^a = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x, \forall \ a \in R$
- $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e \ \ (a>0, a\neq 1)$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \ a>0$
- $\lim_{x\to\infty} \frac{\log x}{x^m} = 0 \ \ (m>0)$
- $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$, where m,n are rational numbers
- $\lim_{x\to 0} \frac{(1+x)^m - 1}{x} = m$

SOME THEOREMS ON LIMITS

- Let $f(x)$ and $g(x)$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist finitely, then

(i) $\lim_{x\to a} (f \pm g)(x)$ and $\lim_{x\to a} f(x)g(x)$ exist.

(ii) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists.

However, the converse of any of the above is not necessarily true.

- $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ provided $\lim_{x\to a} g(x)$ exists and lies in the domain of $f(x)$.

METHODS FOR CALCULATING THE LIMITS

1. When the limit is of the form $\lim_{x\to a}(f(x))^{g(x)}$ and $\lim_{x\to a} g(x) = \infty$

Here we discuss two different cases,

(i) When $\lim_{x\to a} f(x) = 1$.

In this case, $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(f(x)-1)}$

(ii) When $\lim_{x\to a} f(x) \neq 1$, but $f(x)$ is positive in the neigbourhood of $x = a$.

In this case we write, $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$

$\Rightarrow \lim_{x\to a} f(x))^{g(x)} = e^{\lim_{x\to a} g(x)\cdot \ln(f(x))}$

2. L'Hospital's Rule for Calculating Limits

Let f and g be two differentiable functions in the neighbourhood of a which satisfy

(i) $g'(x) \neq 0$ for any x in the neighbourhood of a

(ii) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

This is L′ Hospital's Rule for $\frac{0}{0}$ form. However if $\lim_{x\to a} f'(x) = 0$ and $\lim_{x\to a} g'(x) = 0$, then again we can apply the L′ Hospital's rule.

3. Sandwich Theorem

Let f, g and h be three functions such that in the neighbourhood of a, $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.

CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

i.e. L.H.L.=R.H.L.

= Value of the function at a or $\lim_{x\to a} f(x) = f(a)$

Note : $f(x)$ will be discontinuous at $x = a$ in any of the following cases:

(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limits does not exist.

1. Properties of a Continuous Function

- A function $f(x)$, continuous on the interval $[a, b]$ possesses the following properties:

(i) $f(x)$ is bounded on $[a, b]$, *i.e.* there exist m and M such that $m \le f(x) \le M$ for all $x \in [a, b]$. Here m and M are minimum and maximum values of $f(x)$ respectively.

(ii) Let A be any number such that $\min f(x) \le A \le \max f(x)$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = A$. It is called intermediate value theorem.

- If f and g are continuous at $x = a$, then $f + g$, kf and fg are continuous at $x = a$. Moreover, if $g(a) \neq 0$, then f/g is also continuous at $x = a$. Further if g is continuous at a and f is continuous at $g(a)$, then the composition fog is continuous at $x = a$.

DIFFERENTIABILITY

Let $y = f(x)$ be a continuous function at a point $x = a$. It is said to be differentiable at $x = a$ if $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite.

If this limit exists, we call it the derivative of $y = f(x)$ at a and is denoted by $f'(a)$.

RIGHT AND LEFT HAND DERIVATIVES

- Right hand derivative of $f(x)$ at $x = a$ is given by
$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
- Left hand derivative of $f(x)$ at $x = a$ is given by
$$Lf'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

Clearly, $f(x)$ is differentiable at $x = a$ if and only if $Rf'(a) = Lf'(a)$.

Note : (i) If a function $f(x)$ is differentiable at $x = a$ then, it is also continuous at $x = a$. But if a function is continuous at a point, it is not necessarily differentiable at that point.

(ii) Geometrically, we interpret $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ as the slope of the graph at the point $(a, f(a))$. The line through $(a, f(a))$ which has this slope is called the tangent line at $(a, f(a))$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point. In other words, a function is not differentiable at a point where the curve suddenly changes direction (corner point) or the tangent is vertical at some point.

PROBLEMS

1. Which one of the following is not always true?

(a) If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.

(b) If $f(x)$ is continuous at $x = a$, then it is differentiable at $x = a$.

(c) If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $f(x) + g(x)$ is also differentiable at $x = a$.

(d) If a function $f(x)$ is continuous at $x = a$, then $\lim_{x\to a} f(x)$ exists.

2. The function $f(x) = \begin{cases} 2x+1, & x \in Q \\ (x-1)^2 + 4, & x \notin Q \end{cases}$ is

(a) continuous nowhere

(b) differentiable nowhere

(c) continuous but not differentiable exactly at one point

(d) differentiable and continuous only at one point and discontinuous elsewhere

3. The value of $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\int_{\pi/2}^{x} t\,dt}{\sin(2x-\pi)}$ is

(a) ∞ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{8}$

4. The value of p for which the function

$$f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\frac{x}{p}\log\left[1+\frac{x^2}{3}\right]}, & x\neq 0\\ 12(\log 4)^3, & x=0\end{cases}$$

may be continuous at $x = 0$, is

(a) 1 (b) 2
(c) 3 (d) none of these

5. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC and p be the perimeter of ABC then $\lim\limits_{h\to 0}\dfrac{\Delta}{p^3}$ equals (where Δ is the area of the triangle)

(a) $\dfrac{1}{32r}$ (b) $\dfrac{1}{64r}$
(c) $\dfrac{1}{128r}$ (d) none of these

6. The function $f(x) = [x]^2 - [x^2]$, (where [.] is the greatest integer function), is discontinuous at

(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

7. If $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then

(a) $b = 0, c = 0, a$ is any real
(b) $a = 0, b = 0, c$ is any real
(c) $c = 0, a = 0, b$ is any real
(d) $a + b = 0$

8. Let $f : R \to R$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Which of the following is true?

(a) $f(x) \geq 1$ for all $x \in R$.
(b) $f(x)$ is not differentiable at $x = 1$.
(c) $f(x)$ is differentiable everywhere.
(d) $f(x)$ is not differentiable at $x = 0$.

9. Let α and β be the roots of the equation, $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$. If $\lim\limits_{x\to m}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$, then

(a) $a > 0$ and $m > 1$
(b) $a < 0$ and $m < 1$
(c) $a < 0$ and $\alpha < m < \beta$
(d) $\dfrac{|a|}{a}=1$ and $m > \alpha$

10. The function $f : R - \{0\} \to R$, $f(x)=\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining $f(0)$ as

(a) 2 (b) –1 (c) 0 (d) 1

11. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$. Then the value of $f^2(4) + g^2(4)$ is

(a) 8 (b) 16 (c) 32 (d) 64

12. The number of values of a for the which the function $f(x) = (x + 1)|x - a|$ is differentiable $\forall\, x \in R$, is

(a) 0 (b) 1
(c) 2 (d) more than 2

13. If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(a) $a = 1, b = 1$ (b) $a = 1, b = 2$
(c) $a = 1, b = -2$ (d) none of these

14. $\lim\limits_{x\to\infty}\left(\dfrac{1}{e}-\dfrac{x}{1+x}\right)^x$ is equal to

(a) $\dfrac{e}{1-e}$ (b) 0
(c) $\dfrac{e}{e^{1-e}}$ (d) does not exist

15. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$ is equal to

(a) 0 (b) ∞
(c) $\dfrac{1}{2}$ (d) none of these

16. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?

I. f is continuous on the closed interval $[a, b]$.
II. f is bounded on the open interval (a, b).
III. If $a < a_1 < b_1 < b$ and $f(a_1) < 0 < f(b_1)$, then there is a number c such that $a_1 < c < b_1$ and $f(c)=0$.

(a) I and II only (b) I and III only
(c) II and III only (d) only III

17. $\lim\limits_{x\to\infty}\dfrac{2\sqrt{x}+3\sqrt[3]{x}+4\sqrt[4]{x}+.....+n\sqrt[n]{x}}{\sqrt{(2x-3)}+\sqrt[3]{(2x-3)}+.....+\sqrt[n]{(2x-3)}}$ is equal to

(a) 1 (b) ∞
(c) $\sqrt{2}$ (d) none of these

18. If $f(x)=\lim\limits_{n\to\infty}\left(x^{\frac{1}{n}}-1\right)n$, then for $x, y>0, f(xy)$ is equal to

(a) $f(x)\,f(y)$ (b) $f(x)+f(y)$
(c) $f(x)-f(y)$ (d) none of these

19. The value of $\lim\limits_{n\to\infty}\left(\sum\limits_{r=1}^{n}\left(\dfrac{2r+5}{r^2+r}\right)\left(\dfrac{3}{5}\right)^{r+1}\right)$ is equal to

(a) 0 (b) $\dfrac{9}{5}$
(c) $\dfrac{3}{5}$ (d) none of these

20. $\lim\limits_{x\to0}\left\{(1+x)^{\frac{2}{x}}\right\}$ (where $\{\cdot\}$ denotes the fractional part of x) is equal to

(a) e^2-7 (b) e^2-8
(c) e^2-6 (d) none of these

21. $\lim\limits_{x\to0}\dfrac{\cos(\tan x)-\cos x}{x^4}$ is equal to

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) 1

22. Among (i) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{x}{\sin x}\right)$ and (ii) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{\sin x}{x}\right)$

(a) (i) exists, (ii) does not exist
(b) (i) does not exist, (ii) exists
(c) both (i) and (ii) exist
(d) neither (i) nor (ii) exists

23. $\lim\limits_{x\to1}\dfrac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}, n\in N=$

(a) ${}^{2n}P_n$ (b) ${}^{2n}C_n$
(c) $2n!$ (d) none of these

24. If $\lim\limits_{x\to0}(x^{-3}\sin3x+ax^{-2}+b)$ exists and is equal to 0, then

(a) $a=-3$ and $b=\dfrac{9}{2}$ (b) $a=3$ and $b=\dfrac{9}{2}$
(c) $a=-3$ and $b=\dfrac{-9}{2}$ (d) $a=3$ and $b=\dfrac{-9}{2}$

25. If $f(x)=\begin{cases}3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}; \dfrac{-1}{2}<x<0\\ \beta \qquad\qquad ; x=0\\ 3\left(1+\left|\dfrac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}}; 0<x<\dfrac{2}{3}\end{cases}$

is a continuous function at $x=0$, then the ordered pair (α, β) is equal to

(a) $(2, e^2)$ (b) $\left(2, \dfrac{2}{e^2}\right)$
(c) $(2, 3e^2)$ (d) $\left(2, \dfrac{3}{e^2}\right)$

26. For $x>0$, let $h(x)$

$=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p \text{ and } q>0 \text{ are relatively prime integers}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

then which one of the following does not hold good?
(a) $h(x)$ is discontinuous for all x in $(0, \infty)$
(b) $h(x)$ is continuous for each irrational in $(0, \infty)$
(c) $h(x)$ is discontinuous for each rational in $(0, \infty)$
(d) $h(x)$ is not derivable for all x in $(0, \infty)$.

27. Let $f:(1, \infty)\to(0, \infty)$ be a continuous decreasing function with $\lim\limits_{x\to\infty}\dfrac{f(4x)}{f(8x)}=1$. Then $\lim\limits_{x\to\infty}\dfrac{f(6x)}{f(8x)}$ is equal to

(a) $\dfrac{4}{8}$ (b) $\dfrac{4}{6}$ (c) $\dfrac{6}{8}$ (d) 1

28. The value of $\lim\limits_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^m$ is

(a) 1 (b) e
(c) e^{-1} (d) none of these

29. Given $f(x)=\begin{cases}x \text{ for } |x|\le1\\ 1 \text{ for } |x|>1\end{cases}$,

$$g(x)=\begin{cases}\cos\left(\frac{\pi}{2}x\right) & \text{for } |x|\le 1\\ |x-1| & \text{for } |x|>1\end{cases},$$

$$h(x)=\begin{cases}\frac{(|x|-1)}{\log_a|x|} & \text{for } |x|\neq 1\\ \ln a & \text{for } |x|=1, a>0, a\neq 1\end{cases}$$

If l, m, n denote the number of points of discontinuity of the functions f, g and h in their domain respectively, over R, then (l, m, n) is

(a) $(0, 0, 0)$ (b) $(1, 1, 1)$

(c) $(2, 2, 2)$ (d) $(1, 1, 0)$

30. $\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$ is equal to

(a) $(n!)^n$ (b) $(n!)^{1/n}$

(c) $n!$ (d) $\ln(n!)$

31. If $\lim\limits_{x\to 0^-}\sin^{-1}[\tan x]=l$, then value of $\{l\}$ is equal to (where [.] and {.} denotes greatest integer function and fractional part of a function)

(a) 0 (b) $1-\frac{\pi}{2}$ (c) $\frac{\pi}{2}-1$ (d) $2-\frac{\pi}{2}$

32. If $f(x) = |1 - x|$, then the number of points where $g(x) = \sin^{-1}(f(|x|))$ is non-differentiable, are

(a) 2 (b) 3 (c) 4 (d) 5

33. Let $f(x)=\frac{1}{1+e^{\frac{1}{x}}}$ for $x\neq 0$ and $f(0) = 0$, then

(a) $f(0^+)$ does not exist (b) $f(0^-) = 0$

(c) $f'(0^-) = 1$ (d) $f'(0^+) = 0$

34. Let $f(x)=\lim\limits_{n\to\infty}\frac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5}$. Then the set of values of x for which $f(x) = 0$ is

(a) $|2x|>\sqrt{3}$ (b) $|2x|<\sqrt{3}$

(c) $|2x|\ge\sqrt{3}$ (d) $|2x|\le\sqrt{3}$

35. If $f(x)=\begin{cases}x^n\sin\left(\frac{1}{x^2}\right) & , x\neq 0, n\in I\\ 0 & , x=0\end{cases}$

Then,

(a) $\lim\limits_{x\to 0}f(x)$ exists for $n>1$

(b) $\lim\limits_{x\to 0}f(x)$ exists for $n<0$

(c) $\lim\limits_{x\to 0}f(x)$ does not exists for any value of n

(d) $\lim\limits_{x\to 0}f(x)$ cannot be determined

SOLUTIONS

1. (b) : If $f(x)$ is continuous at $x = a$, then it is not always differentiable at $x = a$.

2. (d) : $y = (x - 1)^2 + 4 = x^2 - 2x + 5$

According to question,

$y = x^2 - 2x + 5 = 2x + 1$

$\Rightarrow x^2 - 4x + 4 = 0 = (x - 2)^2 = 0 \Rightarrow x = 2$

$\therefore$ $f(x)$ is continuous and differentiable at $x = 2$ only.

3. (c) : Let $y=\lim\limits_{x\to\pi/2}\frac{\int_{\pi/2}^{x}t\,dt}{\sin(2x-\pi)}\Rightarrow y=\lim\limits_{x\to\pi/2}\frac{\left[\frac{t^2}{2}\right]_{\pi/2}^{x}}{\sin(2x-\pi)}$

$\therefore y=\lim\limits_{x\to\frac{\pi}{2}}\frac{\left(\frac{x^2}{2}-\frac{\pi^2}{8}\right)}{\sin(2x-\pi)}\Rightarrow y=\lim\limits_{x\to\frac{\pi}{2}}\frac{(4x^2-\pi^2)}{8[\sin(2x-\pi)]}$

$\Rightarrow y=\lim\limits_{x\to\pi/2}\frac{1}{8}\frac{(2x-\pi)(2x+\pi)}{\sin(2x-\pi)}$

$\therefore y=\frac{1}{8}\frac{\lim\limits_{x\to\pi/2}(2x+\pi)}{\lim\limits_{x\to\pi/2}\frac{\sin(2x-\pi)}{(2x-\pi)}}\Rightarrow y=\frac{1}{8}\times 2\pi=\frac{\pi}{4}$

4. (d) : For $f(x)$ to be continuous at $x = 0$, we have

$\lim\limits_{x\to 0}f(x)=f(0)=12(\log 4)^3$ ...(i)

Now, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right)^3\times\frac{\left(\frac{x}{p}\right)}{\left(\sin\frac{x}{p}\right)}\cdot\frac{px^2}{\log\left(1+\frac{1}{3}x^2\right)}$

$=(\log 4)^3\cdot 1\cdot p\cdot\lim\limits_{x\to 0}\left(\frac{x^2}{\frac{1}{3}x^2-\frac{1}{18}x^4+\ldots}\right)=3p\,(\log 4)^3$

$\therefore$ On comparing with (i), we get $p = 4$.

5. (c) : We have, $AB=\sqrt{h^2+2rh-h^2}=\sqrt{2rh}$

$p=2\sqrt{2rh}+2\sqrt{2rh-h^2}=2[\sqrt{2rh}+\sqrt{2rh-h^2}]$

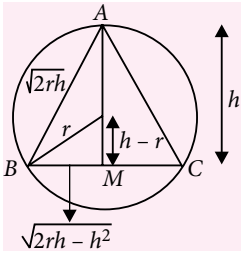
Now, $\Delta=\frac{1}{2}2h\sqrt{2rh-h^2}=h\sqrt{2rh-h^2}$

$\therefore \lim_{h\to 0}\frac{\Delta}{p^3}$

$= \lim_{h\to 0}\frac{h\sqrt{2rh-h^2}}{8[\sqrt{2rh}+\sqrt{2rh-h^2}]^3}$

$\therefore \lim_{h\to 0}\frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}[\sqrt{2r}+\sqrt{2r-h}]^3}$

$= \frac{\sqrt{2r}}{8\cdot 8(2r)(2r)^{1/2}} = \frac{1}{128r}$



6. (d) : Given $f(x) = [x]^2 - [x^2]$

Let $x = n,\ n \in I$

Now, $\lim_{x\to n^+} f(x) = n^2 - n^2 = 0$

and $\lim_{x\to n^-} f(x) = (n-1)^2 - (n^2-1) = -2n+2$

Also, $f(n) = 0$

For the function to be continuous at $x = n$

$-2n + 2 = 0 \Rightarrow n = 1$

Hence, function is discontinuous at all integers except 1.

7. (d) : $f'(0^-) = \lim_{h\to 0}\frac{a\sin h + be^h + ch^3 - b}{-h} = -(a+b)$

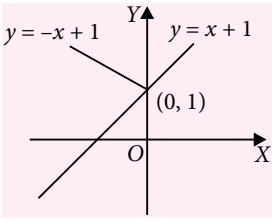
$f'(0^+) = \lim_{h\to 0}\frac{a\sin h + be^h + ch^3 - b}{h}$

$= \frac{a(\sin h)}{h} + \frac{b(e^h-1)}{h} + \frac{ch^3}{h} = a+b$

$\because$ $f(x)$ is differentiable at $x = 0$

$\therefore a + b = -(a + b) \Rightarrow (a + b) = 0.$

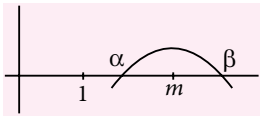
8. (c) : Given, $f(x) = \min\{x + 1, |x| + 1\}$



$\therefore$ $f(x) = x + 1\ \forall\ x \in R$

9. (c) : Given, $\lim_{x\to m}\frac{|ax^2+bx+c|}{ax^2+bx+c} = 1$

$= \lim_{x\to m}\frac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)}$



If $a < 0$ and $\alpha < m < \beta$, then

$a\ (m - \alpha)\ (m - \beta) > 0$

$\Rightarrow$ Limit is 1.

10. (d) : $\lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{e^{2x}-1}\right) = \lim_{x\to 0}\frac{e^{2x}-1-2x}{x(e^{2x}-1)}$

$= \lim_{x\to 0}\frac{\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+.....\right)-1-2x\right\}}{x\left\{\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+.....\right)-1\right\}} = 1 = f(0)$

11. (c) : Given, $f'(x) = g(x)$

$\therefore$ $f''(x) = g'(x) = -f(x) \quad \Rightarrow f''(x) + f(x) = 0$

Solution of the differential equation is given by

$f(x) = c_1\cos x + c_2\sin x$ ...(i)

$\Rightarrow$ $f'(x) = -c_1\sin x + c_2\cos x = g(x)$ ...(ii)

Now, $f(2) = 4 \Rightarrow c_1\cos 2 + c_2\sin 2 = 4$ ...(iii)

$f'(2) = 4 \Rightarrow -c_1\sin 2 + c_2\cos 2 = 4$ ...(iv)

Squaring and adding, (iii) and (iv), we get

$c_1^2 + c_2^2 = 32$ ...(v)

Now, $f(4) = c_1\cos 4 + c_2\sin 4$ ...(vi)

$g(4) = -c_1\sin 4 + c_2\cos 4$ ...(vii)

Squaring and adding, (vi) and (vii), we get

$f^2(4) + g^2(4) = c_1^2 + c_2^2 = 32$ ...(viii)

$\therefore$ $f^2(4) + g^2(4) = 32$

12. (b) : If function is differentiable in R, then it must be differentiable at $x_0 = a$.

$f'(a^+) = \lim_{h\to 0}\frac{(a+h+1)|h|-0}{h} = a+1$

$f'(a^-) = \lim_{h\to 0}\frac{(a-h+1)|-h|}{-h} = -(a+1)$

$\therefore$ $f'(a^+) = f'(a^-) \Rightarrow a = -1$

13. (c) : Given, $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1} - (ax+b)\right) = 2$

$\Rightarrow \lim_{x\to\infty}\frac{x^3(1-a) - bx^2 - ax + (1-b)}{x^2+1} = 2$

$\Rightarrow$ $1 - a = 0$ and $-b = 2 \Rightarrow a = 1, b = -2.$

14. (d) : $\lim_{x\to\infty}\left(\frac{1}{e} - \frac{x}{1+x}\right)^x = \lim_{x\to\infty}\left(\frac{1}{e} - \frac{1}{\frac{1}{x}+1}\right)^x$

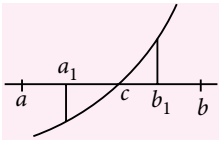
$= \left(\frac{1}{e} - 1\right)^\infty$ = (some negative value)$^\infty$

which is not defined.

15. (c) : $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}}$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{2}{x}-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}}\right)^{\frac{2+1/x}{2-1/x}} = \frac{1}{2}$$

16. (d) : I and II are false. The function $f(x) = 1/x$, $0 < x < 1$, is a counter example.



Statement III is true.

Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.

17. (c) : Since highest degree of x is $\frac{1}{2}$, divide numerator and denominator by $\sqrt{x}$, then value of limit becomes $\frac{2}{\sqrt{2}}$ or $\sqrt{2}$.

18. (b) : $\lim\limits_{n\to\infty} n\left(x^{\frac{1}{n}}-1\right) = \lim\limits_{n\to\infty}\dfrac{x^{1/n}-1}{\frac{1}{n}}$

$= \lim\limits_{m\to 0}\dfrac{x^m-1}{m}\left(\text{where } \frac{1}{n} \text{ is replaced by } m\right) = \ln x$

Now, $f(xy) = \ln(xy) = \ln x + \ln y = f(x) + f(y)$.

19. (b) : Let $T_r = \left(\dfrac{2r+5}{r(r+1)}\right)\left(\dfrac{3}{5}\right)^{r+1}$

$$= \left(\frac{5(r+1)-3r}{r(r+1)}\right)\left(\frac{3}{5}\right)^{r+1} = 3\left[\frac{1}{r}\left(\frac{3}{5}\right)^{r} - \frac{1}{r+1}\left(\frac{3}{5}\right)^{r+1}\right]$$

$= 3(v_r - v_{r+1})$

So, $S_n = \sum\limits_{r=1}^{n} T_r = 3\left[\sum\limits_{r=1}^{n} v_r - \sum\limits_{r=1}^{n} v_{r+1}\right]$

$$\Rightarrow \quad S_n = 3(v_1 - v_{n+1}) = \frac{9(n+1)5^n - 3^{n+2}}{(n+1)\,5^{n+1}}$$

So, $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\dfrac{9(n+1)5^n - 3^{n+2}}{(n+1)\,5^{n+1}} = \frac{9}{5}$.

20. (a) : $\{(1+x)^{2/x}\} = (1+x)^{2/x} - [(1+x)^{2/x}]$

Now, $\lim\limits_{x\to 0}(1+x)^{2/x} = e^2$

$\therefore \quad \lim\limits_{x\to 0}\{(1+x)^{2/x}\} = e^2 - [e^2] = e^2 - 7.$

21. (b) : Since, $\cos(\tan x) - \cos x$

$$= 2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)$$

$$\therefore \quad \lim_{x\to 0}\frac{\cos(\tan x)-\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{x+\tan x}{2}\right)\sin\left(\frac{x-\tan x}{2}\right)}{x^4\left(\frac{x+\tan x}{2}\right)\left(\frac{x-\tan x}{2}\right)}\left(\frac{x^2-\tan^2 x}{4}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{x^2-\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots\right)^2}{x^4}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\left(1-\left(1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots\ldots\right)^2\right) = \frac{1}{3}$$

22. (a) : (i) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{x}{\sin x}\right) = \sec^{-1}\left(\dfrac{\infty}{\sin\infty}\right)$

$$= \sec^{-1}\left(\frac{\infty}{\text{any value between } -1 \text{ to } 1}\right)$$

$$= \sec^{-1}(\pm\infty) = \frac{\pi}{2}$$

(ii) $\lim\limits_{x\to\infty}\sec^{-1}\left(\dfrac{\sin x}{x}\right) = \sec^{-1}\left(\dfrac{\sin\infty}{\infty}\right)$

$$= \sec^{-1}\left(\frac{\text{any value between } -1 \text{ to } 1}{\infty}\right)$$

$= \sec^{-1}0 =$ not defined

Hence (i) exists but (ii) does not exist.

23. (b) : $\lim\limits_{x\to 1}\dfrac{(1-x)(1-x^2).....(1-x^{2n})}{\{(1-x)(1-x^2)....(1-x^n)\}^2}$

$$= \lim_{x\to 1} \frac{\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^{2n}}{1-x}\right)}{\left(\left(\frac{1-x}{1-x}\right)\left(\frac{1-x^2}{1-x}\right)\cdots\left(\frac{1-x^{n}}{1-x}\right)\right)^2}$$

$$= \frac{1\times 2\times 3\ldots..(2n)}{(1\times 2\times 3\times \ldots.\times n)^2} = \frac{(2n)!}{n!n!} = {}^{2n}C_n$$

24. (a) : We have,

$$\lim_{x\to 0} \frac{\sin 3x}{x^3} + \frac{a}{x^2} + b = \lim_{x\to 0} \frac{\sin 3x + ax + bx^3}{x^3}$$

$$= \lim_{x\to 0} \frac{3\frac{\sin 3x}{3x} + a + bx^2}{x^2}$$

For existence of limit, $3 + a = 0 \Rightarrow a = -3$.

Now, $\lim_{x\to 0} \frac{\sin 3x - 3x + bx^3}{x^3} = 0$

$$= 27 \lim_{t\to 0} \frac{\sin t - t}{t^3} + b = 0 \qquad (\text{Put } 3x = t)$$

$$= \frac{-27}{6} + b = 0 \Rightarrow b = \frac{9}{2}.$$

25. (c) : We have,

$$f(x) = \begin{cases} 3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}; & \frac{-1}{2} < x < 0 \\ \beta & ; \ x = 0 \\ 3\left(1+\left|\frac{\sin x}{3}\right|\right)^{\frac{6}{|\sin x|}}; & 0 < x < \frac{2}{3} \end{cases}$$

Now, $\text{R.H.L} = \lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} 3\left(1+\frac{|\sin x|}{3}\right)^{\frac{6}{|\sin x|}} = 3e^2 \qquad \ldots(i)$$

Similarly, $\text{L.H.L} = \lim_{x\to 0^-} f(x) = 3e^{\alpha}$ ...(ii)

$\because$ $f(x)$ is continuous at $x = 0$

$\therefore$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$\Rightarrow$ $3e^{\alpha} = 3e^2 = \beta$.

So, $\alpha = 2$, $\beta = 3e^2$.

26. (a) : Let $x = \frac{2}{3}$ which is rational

$$\Rightarrow \quad h\left(\frac{2}{3}\right) = \frac{1}{3}$$

$$\lim_{t\to 0} h\left(\frac{2}{3}+t\right) = 0 \quad \Rightarrow \text{ Discontinuous at } x \in Q$$

Let $x = \sqrt{2} \notin Q \Rightarrow h(\sqrt{2}) = 0$

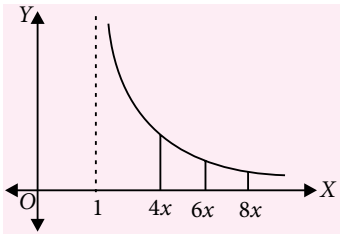
As $\sqrt{2} = 1.414213562$

$\therefore$ $\lim_{x\to\sqrt{2}^+} h(x) = \lim_{x\to\sqrt{2}^-} h(x) = h(\sqrt{2}) = 0$

Hence, $h(x)$ is continuous for all irrationals.

27. (d) : As $f(x)$ is a continuous decreasing positive function on $(1, \infty)$, so $0 < f(8x) < f(6x) < f(4x)$

$$\Rightarrow \quad 0 < 1 < \frac{f(6x)}{f(8x)} < \frac{f(4x)}{f(8x)}$$



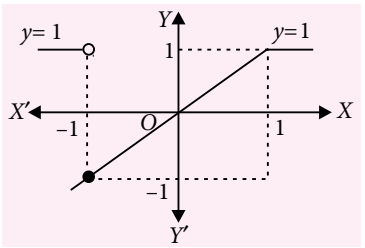
$$\Rightarrow \quad \lim_{x\to\infty} 1 \le \lim_{x\to\infty} \frac{f(6x)}{f(8x)} \le \lim_{x\to\infty} \frac{f(4x)}{f(8x)}$$

$\therefore$ By squeeze theorem, $\lim_{x\to\infty} \frac{f(6x)}{f(8x)} = 1$

28. (a) : $\lim_{m\to\infty} \left(\cos\frac{x}{m}\right)^m = \lim_{m\to\infty} \left[1-\left(1-\cos\frac{x}{m}\right)\right]^m$

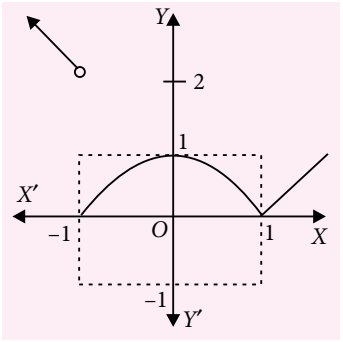
$$= \lim_{m\to\infty} \left[1-2\sin^2\frac{x}{2m}\right]^m = e^{m\to\infty \lim \left(1-2\sin^2\frac{x}{2m}\right)m} = 1.$$

29. (d) : $f(x) = \begin{cases} x \text{ for } -1 \le x \le 1 \\ 1 \text{ for } x > 1 \text{ or } x < -1 \end{cases}$



Clearly, $f(x)$ is discontinuous at $x = -1$

$$g(x) = \begin{cases} \cos\frac{\pi x}{2} & \text{for } -1 \le x \le 1 \\ x-1 & \text{for } x > 1 \\ 1-x & \text{for } x < -1 \end{cases}$$



Clearly $g(x)$ is discontinuous at $x = -1$.

Now, $h(x) = \begin{cases} \dfrac{x-1}{\log_a x}, & x>0,\ x\neq 1 \\ \dfrac{-x-1}{\log_a(-x)}, & x<0,\ x\neq 1 \\ \ln a, & |x|=1,\ a\neq 1 \end{cases}$

Clearly, $h(x)$ is continuous for all x including $x = 1$ or -1.

$\because$ $x = 0$ is not in the domain

$\therefore$ There is no point of discontinuity.

30. (b) : We have, $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+....+n^x}{n}\right)^{\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\left(\frac{1^x-1}{n}+\frac{2^x-1}{n}+....+\frac{n^x-1}{n}\right)\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\frac{1}{n}\left\{\frac{1^x-1}{x}+\frac{2^x-1}{x}+....+\frac{n^x-1}{x}\right\}}$

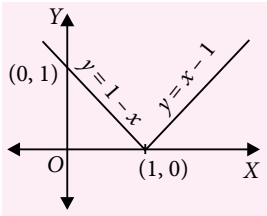
$= e^{\frac{1}{n}[\log 1+\log 2+....+\log n]} = e^{\frac{1}{n}(\log n!)} = e^{(\log n!)^{1/n}} = (n!)^{1/n}$

31. (d) : $\lim\limits_{x\to 0^-}\sin^{-1}[\tan x] = -\dfrac{\pi}{2}$;

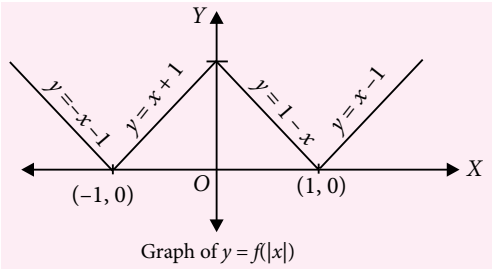
Hence, $\left\{-\dfrac{\pi}{2}\right\} = \left\{2-\dfrac{\pi}{2}-2\right\} = 2-\dfrac{\pi}{2}$

where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer and fractional part function.

32. (d) :



Graph of $y = f(x) = |x - 1|$



Graph of $y = f(|x|)$

Clearly, the domain of $g(x) = \sin^{-1}(f(|x|))$ is $[-2, 2]$

$\Rightarrow$ $g(x)$ is non-differentiable at $x = -2, -1, 0, 1, 2$.

$\therefore$ Number of non-derivable points are 5.

33. (d) : $f(x) = \begin{cases} \dfrac{1}{1+e^{1/x}}, & x\neq 0 \\ 0, & x=0 \end{cases}$

$\therefore$ $f'(0^+) = \lim\limits_{h\to 0}\dfrac{\frac{1}{1+e^{1/h}}}{h} = \lim\limits_{h\to 0}\dfrac{1/h}{1+e^{1/h}}$

$= \lim\limits_{h\to 0}\dfrac{-1/h^2}{e^{1/h}(-1/h^2)} = 0$

Now, $f'(0^-) = \lim\limits_{h\to 0}\dfrac{\frac{1}{1+e^{-1/h}}}{-h} \to -\infty$;

Also, $f(0^+) = \lim\limits_{h\to 0}\dfrac{1}{1+e^{1/h}} = 0$; $f(0^-) = \lim\limits_{h\to 0}\dfrac{1}{1+e^{-1/h}} = 1$

34. (a) : Given $f(x) = \lim\limits_{n\to\infty}\dfrac{1}{\left(\frac{3}{\pi}\tan^{-1}2x\right)^{2n}+5} = 0$

$\Rightarrow \lim\limits_{n\to\infty}\left[\left(\dfrac{3}{\pi}\tan^{-1}2x\right)^2\right]^n \to \infty \Rightarrow \left(\dfrac{3}{\pi}\tan^{-1}2x\right)^2 > 1$

$\Rightarrow \left|\tan^{-1}2x\right| > \dfrac{\pi}{3}$

$\Rightarrow \tan^{-1}2x < -\dfrac{\pi}{3}$ or $\tan^{-1}2x > \dfrac{\pi}{3}$

$\Rightarrow$ $2x < -\sqrt{3}$ or $2x > \sqrt{3}$ $\Rightarrow$ $|2x| > \sqrt{3}$

35. (a) : For $n > 1$,

$\lim\limits_{x\to 0} x^n \sin\left(\dfrac{1}{x^2}\right) = 0 \times (\text{any value between } -1 \text{ to } 1) = 0$

For $n < 0$,

$\lim\limits_{x\to 0} x^n \sin\left(\dfrac{1}{x^2}\right) = \infty \times (\text{any value between } -1 \text{ to } 1) = \infty$

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE Main & Advanced Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE Main & Advanced. In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. BC is latus rectum of a parabola $y^2 = 4ax$ and A is its vertex. The minimum length of projection of BC on a tangent drawn in portion BAC is

(a) $\sqrt{2}a$ (b) $2\sqrt{2}a$ (c) $2a$ (d) $3\sqrt{2}a$

2. Least value of the expression

$$\frac{1}{2bx-(x^2+b^2+\sin^2 x)}, x \in [-1, 0], b \in [2, 3] \text{ is}$$

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$

(c) $-\frac{1}{8+\sin^2 1}$ (d) none of these

3. If in a right angled triangle ABC, $4\sin A \cos B - 1 = 0$ and $\tan A$ is real, then

(a) angles are in A.P. (b) angles are in G.P.
(c) angles are in H.P. (d) none of these

4. If $|z| = 2$ and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{z-2}{z+2}$, then z_1, z_2, z_3 will be vertices of a/an

(a) equilateral triangle
(b) acute angled triangle
(c) right angled triangle
(d) none of these

5. If $a^2 + b^2 - c^2 - 2ab = 0$, then the point(s) of concurrency of family of straight lines $ax + by + c = 0$ lie(s) on the line

(a) $y = x$ (b) $y = x + 1$
(c) $y = -x$ (d) $x + y = 1$

6. If $f''(x) > 0 \ \forall \ x \in R, f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4), 0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

(c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

7. A vector $\vec{r}$ is equally inclined with the vectors $\vec{a} = \cos\theta \hat{i} + \sin\theta \hat{j}$, $\vec{b} = -\sin\theta \hat{i} + \cos\theta \hat{j}$ and $\vec{c} = \hat{k}$, then angle between $\vec{r}$ and $\vec{a}$ is

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{2}$

8. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then one of the roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ is

(a) c (b) $d - c$ (c) $2c$ (d) $2d$

9. Let f be a differentiable function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y \ (\neq 0) \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x)$ is equal to

(a) $2f(x)$ (b) $\frac{f(x)}{x}$

(c) $2xf(x)$ (d) $2\frac{f(x)}{x}$

10. The number of solutions of the equation $\cos^{-1} x + \cos^{-1}\sqrt{1-x^2} = \pi$ is

(a) 1 (b) 2
(c) 0 (d) none of these

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SOLUTIONS

1. (b) : Let tangent at $P(at^2, 2at)$ make an angle θ with x-axis, then $\tan\theta = \frac{1}{t}$.

Projection of BC on tangent $= BC\sin\theta = \frac{4a}{\sqrt{1+t^2}} \geq 2a\sqrt{2}$ (as $-1 \leq t \leq 1$).

2. (b) : Expression will be least if $2bx - (x^2 + b^2 + \sin^2 x)$ is maximum or $x^2 + b^2 + \sin^2 x - 2bx$ is minimum *i.e.*, $(x - b)^2 + \sin^2 x$ is minimum.

Now $|x - b|$ and $|\sin x|$ are minimum if $x = 0$, $b = 2$.

So, least value is $-\frac{1}{4}$.

3. (a) : Since, $4\sin A\cos B = 1$, so A and B can not be $\frac{\pi}{2}$ [As if $B = \frac{\pi}{2}$, then $\cos B = 0$ and if $A = \frac{\pi}{2}$, $\tan A$ is not defined]

$$C = \frac{\pi}{2},\ B = \frac{\pi}{2} - A \Rightarrow 4\sin A\cos\left(\frac{\pi}{2} - A\right) = 1$$

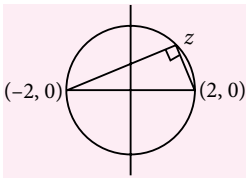
$$\Rightarrow \sin^2 A = \frac{1}{4} \Rightarrow \sin A = \frac{1}{2} \Rightarrow A = \frac{\pi}{6} \text{ and } B = \frac{\pi}{3}$$

So, angles are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, which are in A.P.

4. (c) : Clearly $\arg\left(\frac{z-2}{z+2}\right) = \pm\frac{\pi}{2}$

$$\Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \pm\frac{\pi}{2}$$

So z_1, z_2, z_3 will be the vertices of a right angled triangle.



5. (c) : Given, $(a - b)^2 - c^2 = 0$

$\Rightarrow (a - b - c)(a - b + c) = 0$

If $a - b = c \Rightarrow ax + by + a - b = 0$

$\Rightarrow (x + 1)a + b(y - 1) = 0$

$\Rightarrow x = -1,\ y = 1$

If $-a + b = c \Rightarrow ax + by + b - a = 0$

$\Rightarrow (x - 1)a + (y + 1)b = 0$

$$\Rightarrow (x-1) + (y+1)\frac{b}{a} = 0 \Rightarrow x = 1, y = -1$$

Equation of line passing through both points $(-1, 1)$ and $(1, -1)$ is $y = -x$.

6. (d) : $g'(x) = f'((\tan x - 1)^2 + 3)(2\tan x - 2)\sec^2 x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is increasing.

$$\text{So, } f'((\tan x - 1)^2 + 3) > f'(3) = 0\ \forall x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

$$\text{Also, } (\tan x - 1) > 0\ \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

So, $g(x)$ is increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

7. (c) : Since $\vec{a}\cdot\vec{b} = 0 = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular.

Now $\vec{r} = t(\vec{a} + \vec{b} + \vec{c})$

$$= t((\cos\theta - \sin\theta)\hat{i} + (\cos\theta + \sin\theta)\hat{j} + \hat{k})$$

Let angle between $\vec{r}$ and $\vec{a}$ be α, then

$$\cos\alpha = \frac{\vec{r}\cdot\vec{a}}{|\vec{r}||\vec{a}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

8. (b) : Let $f(x) = x^2 + ax + b$, then $x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$

Thus, roots of $f(x + c) = 0$ will be $0, (d - c)$.

9. (d) : We have $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

Replacing x and y both by 1, $f(1) = \frac{f(1)}{f(1)} = 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\lim_{h\to 0}\left\{\frac{\frac{f(x+h)}{f(x)} - 1}{h}\right\} = f(x)\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right) - 1}{h}$$

$$= \frac{f(x)}{x}\lim_{h\to 0}\frac{f\left(1 + \frac{h}{x}\right) - f(1)}{h/x}$$

$$= \frac{f(x)}{x}f'(1) = 2\frac{f(x)}{x} \quad (\because\ f'(1) = 2).$$

10. (a) : $\cos^{-1}\sqrt{1-x^2} = \pi - \cos^{-1}x = \cos^{-1}(-x)$

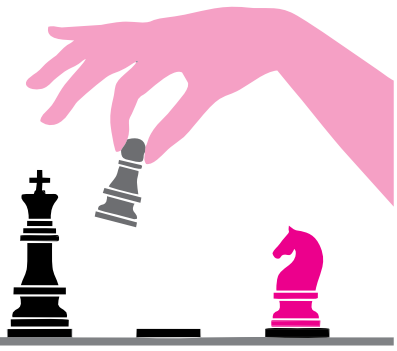
$$\Rightarrow \sqrt{1-x^2} = -x \Rightarrow x < 0$$

On squaring we get, $1 - x^2 = x^2 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Since $x < 0$, $x = -\frac{1}{\sqrt{2}}$.

Challenging PROBLEMS

Integral Calculus



1. $\lim_{n\to\infty}\left[\sin\frac{n}{n^2+1^2}+\sin\frac{n}{n^2+2^2}+\ldots+\sin\frac{n}{n^2+n^2}\right]=$

(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 2π

2. Value of $\lim_{n\to\infty}\left[\frac{\sin\frac{\pi}{n+1}}{1}+\frac{\sin\frac{2\pi}{n+1}}{2}+\ldots+\frac{\sin\frac{n\pi}{n+1}}{n}\right]$ is

(a) positive (b) negative
(c) zero (d) data not sufficient

3. For $k\geq 0$, $\lim_{n\to\infty}\frac{1^k+3^k+\ldots.+(2n-1)^k}{n^{k+1}}=$

(a) $\frac{2^k}{k}$ (b) $\frac{2^k}{k+1}$ (c) $\frac{2^{k+1}}{k}$ (d) $\frac{2^{k+1}}{k+1}$

4. Let f be continuous on $[a, b]$ and let $\int_a^b f(x)dx=0$ then there is a $\theta\in(a, b)$, $(a>0)$ such that

(a) $\int_a^\theta f(x)dx=f(\theta)$ (b) $\int_a^\theta f(x)dx=\theta f(\theta)$
(c) both (a) and (b) (d) none of these

5. $\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}(\log k)^2-\left(\frac{1}{n}\sum_{k=1}^{n}\log k\right)^2\right]=$

(a) 1 (b) 2 (c) 1/2 (d) e

6. If $\int\frac{x+\sin x-\cos x-1}{x+e^x+\sin x}dx=f(x)+\log(g(x))+c$, then $g(0)-f(0)=$

(a) 0 (b) 1 (c) -1 (d) e

7. If $\int\frac{x^4+1}{x^6+1}dx=\tan^{-1}x+\lambda f(x)+C$, then value of constant λ is

(a) 1 (b) 1/2 (c) 1/3 (d) 1/4

8. For a positive integer m,

$\int\frac{x^m}{1+x+\frac{x^2}{2!}+\ldots+\frac{x^m}{m!}}dx=m!x-m!\log(f(x))+C$, then $f(x)=$

(a) $1+x+\frac{x^2}{2!}+\ldots+\frac{x^m}{m!}$ (b) $x+\frac{x^2}{2!}+\ldots+\frac{x^m}{m!}$

(c) $1+x+\frac{x^2}{2!}+\ldots+\frac{x^{m+1}}{(m+1)!}$

(d) $x+\frac{x^2}{2!}+\ldots+\frac{x^{m+1}}{(m+1)!}$

9. If $a, b\in R^+$, then $I=\int_a^b\frac{e^{x/a}-e^{b/x}}{x}dx$ equal

(a) 0 (b) e (c) e^{ab} (d) $e^{a/b}$

10. For $a>0$, $\int_0^\infty\frac{\log x}{x^2+a^2}dx=$

(a) 0 (b) $\pi\log a$
(c) $\frac{\pi\log a}{2}$ (d) $\frac{\pi\log a}{2a}$

11. Let $f:[0, 1]\to[0, \infty)$ be a differentiable function with decreasing first derivative and $f(0)=0$, $f'(0)>0$ then the integral $\int_0^1\frac{dx}{1+f^2(x)}$ is

(a) $\leq\frac{f(1)}{f'(1)}$ (b) $<\frac{f(1)}{f'(1)}$
(c) $\geq\frac{f(1)}{f'(1)}$ (d) $>\frac{f(1)}{f'(1)}$

12. Let $f:R\to R$ be a continuous function such that for all $x\in R$, $\int_0^1 f(xt)dt=0$, then $f(2)=$

(a) 0 (b) 1 (c) 1/2 (d) 2

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13. Consider the two integrals I_1 and I_2 where $f: [0, 1] \to R$ be a continuous function,

$I_1 = \int_0^x \left(\int_0^y f(t)\,dt \right) dy$ and $I_2 = \int_0^x (x-y) f(y)\,dy$. Then,

(a) $I_1 = I_2$ (b) $I_1 + I_2 = 0$

(c) $I_1 = 2I_2$ (d) $I_2 = 2I_1$

14. Let $f: [0, 1] \to R$ be a continuous function such that $x f(y) + y f(x) \le 1$ for all $x, y \in [0, 1]$ then $\int_0^1 f(x)\,dx$

(a) $\le \frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $> \frac{\pi}{2}$ (d) $\frac{\pi}{2}$

SOLUTIONS

1. (a) : For $x > 0$, $x - \frac{x^3}{3!} < \sin x < x$

So, $\sum_{r=1}^{n} \frac{n}{n^2+r^2} - \frac{1}{3!}\sum_{r=1}^{n}\left(\frac{n}{n^2+r^2}\right)^3 < \sum_{r=1}^{n} \sin\frac{n}{n^2+r^2} < \sum_{r=1}^{n} \frac{n}{n^2+r^2}$

Sandwich rule and $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

So, given limit = $\pi/4$.

2. (a) : The function, $f(x) = \begin{cases} \frac{\sin x}{x}, & x \in (0, \pi] \\ 1, & x = 0 \end{cases}$

is continuous on $[0, \pi]$ and positive on $[0, \pi]$.

So, $\int_0^{\pi} f(x)dx > 0$

Now, the given limit can be rewritten as

$$\lim_{n\to\infty} \frac{\pi}{n+1}\left[\frac{\sin\frac{\pi}{n+1}}{\frac{\pi}{n+1}} + \frac{\sin\frac{2\pi}{n+1}}{\frac{2\pi}{n+1}} + ... + \frac{\sin\frac{n\pi}{n+1}}{\frac{n\pi}{n+1}}\right]$$

$$= \pi \cdot \int_0^1 \frac{\sin x}{x}\,dx = \pi \int_0^1 f(x)dx > 0.$$

3. (b) : Rewrite the given sum as

$$\frac{2^k}{n}\left[\left(\frac{1}{2n}\right)^k + \left(\frac{3}{2n}\right)^k + ... + \left(\frac{2n-1}{2n}\right)^k\right]$$

then the given limit is $2^k \cdot \int_0^1 x^k dx = \frac{2^k}{k+1}$

4. (c) : For option (a) : Consider the function $F(x) = e^{-x}\int_a^x f(t)\,dt$, then $F(a) = 0 = F(b)$ from given condition in question.

Now, there exist a $\theta \in (a, b)$ such that $F'(\theta) = 0$

$\Rightarrow \quad e^{-\theta} f(\theta) - e^{-\theta} \cdot \int_a^{\theta} f(t)\,dt = 0$

Similarly, for option (b):

Consider the function $F(x) = \frac{1}{x} \cdot \int_a^x f(t)\,dt$

5. (a) : $a_n = \frac{1}{n}\sum_{k=1}^{n} (\log k)^2 - \left(\frac{1}{n}\sum_{k=1}^{n} \log k\right)^2$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n-1} \left(\log\left(\frac{k}{n}\right)\right)^2 - \left(\frac{1}{n}\sum_{k=1}^{n-1} \log\left(\frac{k}{n}\right)\right)^2$$

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique solution.

9	8							2
	3					9	6	1
	6	1	9		7	8		
		8		3		2		
			7		1			
		9		4		6		
		3	2		4	7	8	
6	9	4					2	
8							4	6

7	4							3
	2					7	9	8
	3	6	9		2	5		
		7		1		9		
			7		4			
		3		5		1		
		4	6		7	8	1	
6	9	1					7	
2							5	6

						3		2
							7	
						8		9

						4		
						5		

8	1						2	3
	6					5	3	7
	4	7	3		9	8		
		6		4		1		
			1		8			
		8		5		7		
		2	9		6	3	5	
4	3	5					1	
6							7	8

6						2		4
						7	8	
						8	4	
						1		
						9		
						5		
						1	3	
						9	4	
						3	8	
						1		
						7	8	9

7	4							3
	2					7	9	8
	3	6	9		2	5		
		7		1		9		
			7		4			
		3		5		1		
		4	6		7	8	1	
6	9	1					7	
2							5	6

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

So, $\lim_{n\to\infty} a_n = \int_0^1 (\log x)^2\,dx - \left(\int_0^1 \log x\,dx\right)^2 = 2-1=1$

6. (b) : Add and subtract e^x in the numerator, then

$$I = \int \frac{(x+e^x+\sin x)-(1+e^x+\cos x)}{x+e^x+\sin x}\,dx$$

$$= x + \log(x + e^x + \sin x)^{-1} + c$$

7. (c) : Let $I = \int \frac{x^4+1}{x^6+1}\,dx = \int \frac{x^4-x^2+1+x^2}{x^6+1}\,dx$

$$= \int \frac{x^4-x^2+1}{x^6+1}\,dx + \int \frac{x^2}{x^6+1}\,dx = I_1 + I_2$$

Now, $I_1 = \int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C_1$

and $I_2 = \int \frac{x^2\,dx}{(x^3)^2+1} = \frac{1}{3}\cdot\int \frac{d(x^3)}{(x^3)^2+1} = \frac{1}{3}\tan^{-1}(x^3) + C_2$

So, $I = \tan^{-1}x + \frac{1}{3}\tan^{-1}(x^3) + C.$

8. (a) : Let $p(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^m}{m!}$

then, $I = \int \frac{m![p(x)-p'(x)]}{p(x)}\,dx = m!\int\left(1-\frac{p'(x)}{p(x)}\right)dx$

$= m!(x - \log p(x)) + C.$

9. (a) : Putting $t = \frac{ab}{x}$, $I = -I$

10. (d) : Let $I = \int_0^\infty \frac{\log x}{x^2+a^2}\,dx$, $x = \frac{a}{t}$

$$= \int_\infty^0 \frac{\log a - \log t}{a^2 + \left(\frac{a}{t}\right)^2}\cdot\left(\frac{-a}{t^2}\right)dt$$

$$= \frac{\log a}{a}\cdot\int_0^\infty \frac{dt}{t^2+1} - \frac{1}{a}\cdot\int_0^\infty \frac{\log t}{t^2+1}\,dt = I_1\left(\frac{\log a}{a}\right) - \frac{1}{a}I_2$$

where $I_1 = \int_0^\infty \frac{dt}{t^2+1} = \tan^{-1}t\Big|_0^\infty = \frac{\pi}{2}$

and $I_2 = \int_0^\infty \frac{\log t}{t^2+1}\,dt$

Putting $t = \frac{1}{y}$ gives $I_2 = -I_2 \Rightarrow I_2 = 0$

Hence, $I = \frac{\pi\log a}{2a}.$

11. (b) : For $x \in [0, 1]$, $f'(x) \geq f'(1)$ (given condition in question)

$$\therefore \quad \frac{f'(1)}{f^2(x)+1} \leq \frac{f'(x)}{f^2(x)+1}$$

On integrating we get,

$$f'(1)\cdot\int_0^1 \frac{dx}{f^2(x)+1} \leq \tan^{-1}(f(1)) - \tan^{-1}(f(0))$$

i.e. $\int_0^1 \frac{dx}{f^2(x)+1} \leq \frac{\tan^{-1}f(1)}{f'(1)} \leq \frac{f(1)}{f'(1)}$

$[\because \tan^{-1}x \leq x \text{ for } x \geq 0]$

For equality to hold, $\tan^{-1}f(1) = f(1)$

i.e., $f(1) = 0$ then $\int_0^1 \frac{dx}{f^2(x)+1} = 0$

which is not possible as this is a strictly positive function. Hence, $\int_0^1 \frac{dx}{f^2(x)+1} < \frac{f(1)}{f'(1)}.$

12. (a) : For any $x \neq 0$, let $xt = y$, then

$$I = \int_0^1 f(xt)dt = \frac{1}{x}\cdot\int_0^x f(y)dy$$

Hence, $\int_0^x f(y)\,dy = 0$ for all $x \in R$

i.e. $\frac{d}{dx}\left(\int_0^x f(y)\,dy\right) = 0 \Rightarrow f(x) = 0$ for all $x \in R$.

13. (a) : In integral I_1, take $g(y) = 1$, $h(y) = \int_0^y f(t)\,dt$ then I_1 becomes $\int_0^x g(y)\,h(y)\,dy$, now use integration by parts to make $I_1 = I_2$.

14. (a) : $I = \int_0^1 f(x)dx = \int_0^{\pi/2} f(\sin t)\cos t\,dt$

$$= \int_0^{\pi/2} f(\cos t)\sin t\,dt$$

Adding, $2I = \int_0^{\pi/2} (f(\sin t)\cos t + f(\cos t)\sin t)\,dt$

$$\leq \int_0^{\pi/2} 1\,dt \text{ [By given condition]}$$

$\therefore \quad I \leq \frac{\pi}{4}.$

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. If $f(x) = x^3 + (a-3)x^2 + \frac{4}{3}(a-4)x + 5$ has a maxima at some $x \in R^-$ and minima at some $x \in R^+$, then find the possible values of a.

(Ramesh Verma, Delhi)

Ans. We have, $f(x) = x^3 + (a-3)x^2 + \frac{4}{3}(a-4)x + 5$

and $f'(x) = 3x^2 + 2(a-3)x + \frac{4}{3}(a-4)$

According to the given condition, $f'(x)$ must vanish at two distinct points say α, β such that $\alpha < 0$ and $\beta > 0$. Then, we have

$f'(x) = 3(x-\alpha)(x-\beta)$

and $f''(x) = 3(2x-\alpha-\beta)$

According to the given condition, α is a point of maxima

$\therefore \quad f''(\alpha) < 0 \quad \Rightarrow 3(\alpha-\beta) < 0$ which is true $\forall\ a$

Also, β is a point of minima

$\therefore \quad f''(\beta) > 0 \quad \Rightarrow 3(\beta-\alpha) > 0$ which is true $\forall\ a$

Now, for $\alpha < 0$ and $\beta > 0$, we have

$\alpha\beta < 0 \quad i.e., \ \frac{4}{9}(a-4) < 0 \quad \Rightarrow a < 4$ or $a \in (-\infty, 4)$.

2. The three normals drawn from a point to parabola $y^2 = 4ax$ meets the axis of parabola in points whose distances from the vertex are in arithmetic progression. Find locus of the point.

(Neetu Singh, Jharkhand)

Ans. Equation of a normal to the parabola $y^2 = 4ax$ is given by $y = mx - 2am - am^3$ ...(i)

$\because$ (i) passes through (h, k),

$\therefore \quad k = mh - 2am - am^3$

i.e., $am^3 + (2a-h)m + k = 0$

If m_1, m_2 and m_3 be the roots of the above equation, then, $m_1 + m_2 + m_3 = 0$ ...(ii)

Also, $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$ ...(iii)

$m_1m_2m_3 = \frac{-k}{a}$ ...(iv)

The normal $y = mx - 2am - am^3$ meets the axis, *i.e.*, $y = 0$ at the point $(2a + am^2, 0)$.

According to the given condition, we have

$(2a + am_1^2) + (2a + am_3^2) = 2(2a + am_2^2)$

i.e., $m_1^2 + m_3^2 = 2m_2^2$ ...(v)

From (ii) and (iii), we have, $m_1m_3 + m_2(m_1 + m_3) = 2 - \frac{h}{a}$

$\Rightarrow \quad m_1m_3 - m_2^2 = 2 - \frac{h}{a}$ ...(vi)

Simplifying (v), we have, $2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3$

$\Rightarrow \quad m_2^2 + 2m_1m_3 = 0$ ...(vii)

Solving equations (vi) and (vii) simultaneously, we have

$m_1m_3 = \frac{2a-h}{3a}$ and $m_2^2 = -2\left(\frac{2a-h}{3a}\right)$

Putting these values in (iv), we have, $(m_1m_2m_3)^2 = \frac{k^2}{a^2}$

$\Rightarrow \quad -2\left(\frac{2a-h}{3a}\right) \times \left(\frac{2a-h}{3a}\right)^2 = \frac{k^2}{a^2}$

$\Rightarrow \quad 2(h-2a)^3 = 27ak^2$

Hence, the required locus is $27\,ay^2 = 2(x-2a)^3$.

3. Given that for each $a \in (0, 1)$,

$$g(a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt \text{ exists. Find } g\left(\frac{1}{2}\right)$$

(Sakshi Gupta, Madhya Pradesh)

Ans. We have, $g(a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$$\therefore \quad g\left(\frac{1}{2}\right) = \lim_{h \to 0^+} \int_h^{1-h} t^{-1/2}(1-t)^{-1/2}\,dt$$

$$\Rightarrow \quad g\left(\frac{1}{2}\right) = \lim_{h \to 0^+} \int_h^{1-h} \frac{1}{\sqrt{t-t^2}}\,dt$$

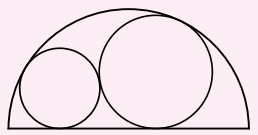
$$\Rightarrow \quad g\left(\frac{1}{2}\right) = \int_0^1 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}}\,dt$$

$$\Rightarrow \quad g\left(\frac{1}{2}\right) = \left[\sin^{-1}\left(\frac{t-\frac{1}{2}}{1/2}\right)\right]_0^1 = \sin^{-1}1 - \sin^{-1}(-1) = \pi.$$

OLYMPIAD CORNER

1. Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$R_1 + R_2 \le 2(\sqrt{2}-1)$.



2. Ten players participate in a ping-pong tournament, in which every two players play against each other exactly once. If player i beats player j, player j beats player k, and player k beats player i, then the set $\{i, j, k\}$ is called a triangle. Let w_i and l_i denote the number of games won and lost, respectively, by the i^{th} player. Suppose that, whenever i beats j, then $l_i + w_j \ge 8$. Prove that there are exactly 40 triangles in the tournament.

3. In an equilateral triangle ABC (of side length 2) consider the incircle T.

(a) Show that for all points P of T,

$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5$.

(b) Show that for all points P of T, it is possible to construct a triangle of sides PA, PB, PC, with area $\sqrt{3}/4$.

4. Are there any nonconstant differentiable functions $f: R \to R$ such that $f(f(f(x))) = f(x) \ge 0$, for all $x \in R$?

5. **(a)** Find all positive integers $p \le q \le r$ satisfying the equation $p + q + r + pq + qr + rp = pqr + 1$.

(b) For each such solution (p, q, r), evaluate $\tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r)$.

6. Let n be a fixed positive integer. Show that for any nonnegative integer k, the equation

$x_1^3 + x_2^3 + \ldots + x_n^3 = y^{3k+2}$

has infinitely many solutions in positive integers x_i and y.

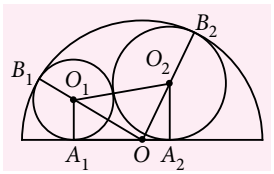
7. Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le u \\ 1-(\sqrt{ux} + \sqrt{(1-u)(1-x)})^2 & \text{if } 0 \le x \le 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows: $u_1 = f(1)$, and $u_n = f(u_{n-1})$ for all $n > 1$.
Show that there exists a positive integer k for which $u_k = 0$.

SOLUTIONS

1. Let O_1, O_2, and O denote the centres of the circles, and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the semicircle, as shown in the diagram. Then $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$, and $O_2A_2 = R_2$. So,



$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2}$$

$$= \sqrt{(R_1+R_2)^2 - (R_1-R_2)^2} = 2\sqrt{R_1R_2}.$$

Also, $OB_1 = 1$, $OB_2 = 1$, $O_1B_1 = R_1$ and $O_2B_2 = R_2$, so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$.
Therefore, $A_1A_2 = OA_1 + OA_2$

$$= \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2}$$

$$= \sqrt{(1-R_1)^2 - R_1^2} + \sqrt{(1-R_2)^2 - R_2^2} = \sqrt{1-2R_1} + \sqrt{1-2R_2}.$$

Thus, $\sqrt{1-2R_1} + \sqrt{1-2R_2} = 2\sqrt{R_1R_2}$.
Squaring, then dividing each term by 2 and rearranging the terms, we get

$$\sqrt{(1-2R_1)(1-2R_2)} = 2R_1R_2 + R_1 + R_2 - 1.$$

Square both sides and simplify:

$8R_1R_2 = (2R_1R_2 + R_1 + R_2)^2$, ...(i)

So, $2\sqrt{2R_1R_2} = 2R_1R_2 + R_1 + R_2$ ...(ii)

Thus, $R_1 + R_2 = 2\sqrt{R_1R_2}(\sqrt{2} - \sqrt{R_1R_2})$

$\leq (R_1 + R_2)(\sqrt{2} - \sqrt{R_1R_2})$, and therefore $\sqrt{R_1R_2} \leq \sqrt{2} - 1$.

Now, consider the function $f(x) = 2x\ (\sqrt{2} - x)$. $f(x)$ is increasing on the interval $(0, 1/\sqrt{2})$ since $f'(x) = 2\sqrt{2} - 4x > 0$ for x in the interval. Since $0 < \sqrt{R_1R_2} \leq \sqrt{2} - 1 < 1/\sqrt{2}$ and $R_1 + R_2 = f(\sqrt{R_1R_2})$ from (ii), $R_1 + R_2$ attains its maximum when $\sqrt{R_1R_2} = \sqrt{2} - 1$.

Hence, $R_1 + R_2 \leq 2(\sqrt{2} - 1)[\sqrt{2} - (\sqrt{2} - 1)] = 2(\sqrt{2} - 1)$.

2. Let P_i denote the i^{th} player $i = 1, 2, \ldots, 10$. Without loss of generality, assume that $w_1 \leq w_2 \leq \ldots \leq w_{10}$. Let $M = \max_{1 \leq i \leq 10} w_i = w_{10}$ and $m = \min_{1 \leq i \leq 10} w_i = w_1$. Since $w_i + l_i = 9$ for all i, the condition $l_i + w_j \geq 8$, whenever P_i beats P_j, is equivalent to

$w_i \leq w_j + 1$. ...(i)

Clearly $\sum_{i=1}^{10} w_i = \binom{10}{2} = 45$. ...(ii)

We first show that

$(w_1, w_2, \ldots, w_{10}) = (4, 4, 4, 4, 4, 5, 5, 5, 5, 5)$

Since P_1 beats m players and loses to $9 - m$ others, we get from (i) and (ii) that $45 \leq m + Mm + (9 - m)(m + 1)$ or $36 \leq (M - m + 9)m$. ...(iii)

Similarly, since P_{10} beats M players and loses to $9 - M$ others, we have $45 \geq M + M(M - 1) + (9 - M)m = M^2 + (9 - M)m$. ...(iv)

Since $M \leq 9$, from (iii) we obtain $36 \leq (18 - m)m$, which implies $m \geq 3$. From (iii) we get $M^2 \leq 45$, *i.e.*, $M \leq 6$. If $M = 6$, then from (iv), $3m \leq 9$ or $m \leq 3$ and so $m = 3$. Similarly, if $m = 3$ then from (iii) $M \geq 6$ and so $M = 6$. Since 10×45, $M \neq m$. Therefore, the only possibilities are $(M, m) = (6, 3)$ and $(5, 4)$.

Note : At this point, Lau listed all 10-tuples $(w_1, w_2, \ldots, w_{10})$ satisfying $(M, m) = (6, 3)$ and (2), and then showed that none were possible. But the case $(M, m) = (6, 3)$ can be eliminated as follows. If $m = w_1 = 3$, then six players beat P_1, so by (i) at least seven of the w_i's (including w_1) are ≤ 4. If $M = w_{10} = 6$, then P_{10} beats six players, so by (i) atleast seven of the w_i's (including w_{10}) are ≥ 5. But since there are only ten w_i's this is impossible.

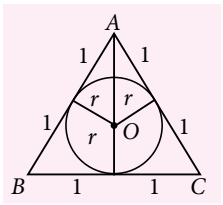
Thus $(M, m) = (5, 4)$, and $(4, 4, 4, 4, 4, 5, 5, 5, 5, 5)$ is the only possible solution. By exhibiting the tournament graphically, one can see that this is indeed a solution. Now, there is a theorem in graph theory which states that the number of transitive triples [3 players P_i, P_j and P_k such that P_i beats P_j, P_j beats P_k, and P_i beats P_k – *Ed.*] in a tournament with n vertices and score sequence $(w_1, w_2, \ldots, w_n)$ is $\sum_{i=1}^{n} w_i (w_i - 1)/2$.

Note : This result is true because the number of such triples is just the number of unordered pairs $\{P_j, P_k\}$ of (distinct) players beaten by P_i, which is $\binom{w_i}{2}$, added up over all P_i.] It follows that, with $n = 10$, the number of triangles (cyclic triples) is

$$\binom{10}{3} - \frac{1}{2}\sum_{i=1}^{n} w_i(w_i - 1) = 120 - \frac{1}{2}(5)(5)(5-1) - \frac{1}{2}(5)(4)(4-1) = 40.$$

This completes the proof.

3. Take O as origin, with the x-axis parallel to BC, and y-axis along OA.



Then the incircle has radius $r = 1/\sqrt{3}$ and A, B, C have coordinates $A\left(0, \frac{2}{\sqrt{3}}\right), B\left(-1, \frac{-1}{\sqrt{3}}\right), C\left(1, \frac{-1}{\sqrt{3}}\right)$.

SAMURAI SUDOKU

SOLUTION - AUGUST 2018

3	9	4	6	5	8	7	1	2
1	2	6	9	7	3	4	8	5
7	8	5	1	4	2	9	3	6
4	7	8	2	1	9	6	5	3
9	5	1	7	3	6	2	4	8
2	6	3	5	8	4	1	7	9
8	4	9	3	6	7	5	2	1
5	3	2	4	9	1	8	6	7
6	1	7	8	2	5	3	9	4

5	3	2	7	1	6	8	4	9
9	7	6	4	2	8	5	1	3
8	1	4	3	9	5	6	7	2
4	8	7	9	5	3	1	2	6
6	2	1	8	4	7	9	3	5
3	9	5	1	6	2	4	8	7
7	6	9	2	8	1	3	5	4
1	4	3	5	7	9	2	6	8
2	5	8	6	3	4	7	9	1

			2	4	5	6	8	7
			9	8	3	5	2	1
			1	7	6	3	4	9

			3	9	1			
			6	7	4			
			8	2	5			

2	6	9	5	4	1	7	3	8
7	1	4	8	3	2	6	5	9
5	8	3	7	9	6	4	1	2
8	3	6	1	2	9	5	4	7
4	5	2	3	8	7	1	9	6
1	9	7	6	5	4	8	2	3
3	2	1	4	6	8	9	7	5
9	7	8	2	1	5	3	6	4
6	4	5	9	7	3	2	8	1

4	5	2	7	9	1	3	8	6
6	3	7	8	4	5	2	9	1
1	9	8	6	2	3	7	4	5
3	4	5	2	1	9	8	6	7
9	2	6	5	3	7	4	1	8
8	7	1	4	6	8	9	5	2
7	8	4	3	5	6	1	2	9
2	6	9	1	8	4	5	3	7
5	1	3	9	7	2	6	8	4

A point on the incircle has coordinates parameterized by θ, $0 \le \theta < 2\pi$ given by $P\left(\frac{1}{\sqrt{3}}\cos\theta, \frac{1}{\sqrt{3}}\sin\theta\right)$.

(a) Let $c = \cos\theta$, $s = \sin\theta$. Then

$AP^2 + BP^2 + CP^2$

$$= \left[\frac{1}{3}c^2 + \frac{1}{3}(2-s)^2\right] + \left[\frac{1}{3}(c+\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right] \left[\frac{1}{3}(c-\sqrt{3})^2 + \frac{1}{3}(s+1)^2\right]$$

$$= \frac{1}{3}[c^2 + 4 - 4s^2 + s^2 + c^2 + 2\sqrt{3}c + 3 + s^2 + 2s + 1 + c^2 - 2\sqrt{3}c + 3 + s^2 + 2s + 1]$$

$$= \frac{1}{3}[3(c^2 + s^2) + 12] = 5$$

since $c^2 + s^2 = 1$.

(b) Now, set $x = AP = \frac{1}{\sqrt{3}}\sqrt{5-4s}$,

$y = BP = \frac{1}{\sqrt{3}}\sqrt{5+2s+2\sqrt{3}c}$, and

$z = CP = \frac{1}{\sqrt{3}}\sqrt{5+2s-2\sqrt{3}c}$.

By reflection and rotational geometry, the distances AP, BP, CP will be a permutation of those obtained when $\pi/6 \le \theta \le \pi/2$, so that $1/\sqrt{3} \le x \le 1$.

Also,

$$y = \frac{1}{\sqrt{3}}\sqrt{5 + 4\left(\frac{1}{2}s + \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5 + 4\cos\left(\theta - \frac{\pi}{6}\right)}$$

so that $\sqrt{7}/\sqrt{3} \le y \le \sqrt{3}$ and

$$z = \frac{1}{\sqrt{3}}\sqrt{5 + 4\left(\frac{1}{2}s - \frac{\sqrt{3}}{2}c\right)} = \frac{1}{\sqrt{3}}\sqrt{5 + 4\sin\left(\theta - \frac{\pi}{3}\right)}$$

so that $\sqrt{5}/\sqrt{3} \le z \le \sqrt{7}/\sqrt{3}$. Thus,

$$(x+y)_{\min} = \frac{1+\sqrt{7}}{\sqrt{3}} > z_{\max} = \frac{\sqrt{7}}{\sqrt{3}}$$

and $$(x+z)_{\min} = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}} > x_{\max} = 1$$

and $$(z+x)_{\min} = \frac{1+\sqrt{5}}{\sqrt{3}} > y_{\max} = \sqrt{3},$$

and x, y, z (*i.e.* AP, BP, CP) can form the sides of a triangle.

From Heron's formula the area, F, of this triangle is given by

$$F^2 = \frac{1}{2}(x+y+z)\frac{1}{2}(-x+y+z)\cdot\frac{1}{2}(x-y+z)\cdot\frac{1}{2}(x+y-z)$$

$$= \frac{1}{16}[(y+z)^2 - x^2][x^2 - (y-z)^2]$$

$$= \frac{1}{16}[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s) + 2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2}] \cdot[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c) + 2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2}]$$

$$= \frac{1}{48}\left[(5+8s) + 2\sqrt{(5+2s)^2 - 12c^2}\right] \left[-(5+8s) + 2\sqrt{(5+2s)^2 - 12c^2}\right]$$

$$= \frac{1}{48}[100 + 80s + 16s^2 - 48c^2 - 25 - 80s - 64s^2]$$

$$= \frac{1}{48}[75 - 48] = \frac{9}{16}, \text{ since } c^2 + s^2 = 1. \text{ Thus } F = 3/4.$$

4. Applying f to both sides of the functional equation

$$f(f(f(x))) = f(x) \ge 0 \qquad \ldots(i)$$

gives $g(g(x)) = g(x)$, where $g(x) = f(f(x))$ for all $x \in R$. Of course g is also a differentiable function on R and $g(x) \ge 0$ for all $x \in R$. Then the range $T = g(R)$ of g is an interval contained in $[0, +\infty)$. Let a be the infimum of T. Since $g(t) = t$ for all $t \in T$ and g is continuous, it follows that $g(a) = a$. Assuming that T has more than one element, choose $\delta > 0$ such that $(a, a + \delta) \subseteq T$. Then $x \in (a - \delta, a)$ implies $g(x) \ge g(a)$ $(= a)$, hence

Therefore, $\frac{g(x)-g(a)}{x-a} \le 0$.

$$g'_L(a) = \lim_{x \to a^-} \frac{g(x)-g(a)}{x-a} \le 0. \qquad \ldots(ii)$$

For $x \in (a, a + \delta)$ we have

$$\frac{g(x)-g(a)}{x-a} = 1.$$

Hence, $$g'_R(a) = \lim_{x \to a^+} \frac{g(x)-g(a)}{x-a} = 1. \qquad \ldots(iii)$$

(ii) and (iii) are contradictory since g is differentiable at a. We are led to the conclusion that T is a single point, *i.e.*, g is a constant function, say $g(x) = c$ for

all $x \in R$. This gives, using (i), that $f(c) = f(x)$ for all $x \in R$, showing that f is a constant function. Thus there is no nonconstant differentiable function satisfying (i).

5. **(a)** There are exactly 3 solutions, given by

$(p, q, r) = (2, 4, 13), (2, 5, 8)$ and $(3, 3, 7)$.

Note first that

$(p-1)(q-1)(r-1) = pqr - (pq + qr + rp) + (p + q + r) - 1$

which becomes, using the given equation,

$(p-1)(q-1)(r-1) = 2(p + q + r - 1)$. ...(i)

If $p \geq 4$, then $4 \leq p \leq q \leq r$ implies

$(p-1)(q-1)(r-1) \geq 9(r-1)$ and $2(p + q + r - 1) \leq 2(3r - 1)$;

and since, $9(r-1) - 2(3r-1) = 3r - 7 > 0$,

(i) cannot hold in this case. Thus $p < 4$. Since $p = 1$ clearly does not satisfy (i), we have $p = 2$ or 3. When $p = 2$, (i) becomes $(q-1)(r-1) = 2(q + r + 1)$ or $(q-3)(r-3) = 10$. Thus $q - 3 = 1$, $r - 3 = 10$ or $q - 3 = 2$, $r - 3 = 5$, yielding two solutions: (2, 4, 13) and (2, 5, 8). When $p = 3$, (i) becomes $(q-1)(r-1) = q + r + 2$ or $(q-2)(r-2) = 5$ which yields the third solution: (3, 3, 7).

(b) The value is $\pi/4$ in all cases. To see this, set $A = \tan^{-1}(1/p)$, $B = \tan^{-1}(1/q)$ and $C = \tan^{-1}(1/r)$

Since $0 < \frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{p} < 1$,

We have, $0 < C \leq B \leq A < \pi/4$

$\therefore \quad 0 < A + B + C < 3\pi/4$.

From the well known formula for $\tan(x + y)$ one easily deduces that for all x, y, z with $x + y + z \neq k\pi + \pi/2$ (where k denotes an integer),

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}$$

$\therefore \quad \tan(A + B + C)$

$$= \frac{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr}}{1 - \left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right)} = \frac{pq + qr + rp - 1}{pqr - (p + q + r)} = 1.$$

Hence, $A + B + C = \pi/4$

$$\therefore \quad \tan^{-1}(1/p) + \tan^{-1}(1/q) + \tan^{-1}(1/r) = \frac{\pi}{4}.$$

6. Since, $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

we see that when $k = 0$,

$$(x_1, x_2, \ldots, x_n; y) = \left(1, 2, \ldots, n; \frac{n(n+1)}{2}\right)$$

is a solution. To see that we can generate infinitely many solutions. In general, set $c = (n(n + 1))/2$ and notice that for all positive integers q, we have:

$(c^k q^{3k+2})^3 + (2c^k q^{3k+2})^3 + \ldots + (nc^k q^{3k+2})^3$

$= c^{3k} q^{3(3k+2)} (1^3 + 2^3 + \ldots + n^3)$

$= c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2$

$= c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}$.

That is, $(x_1, x_2, \ldots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \ldots, nc^k q^{3k+2}; cq^3)$ is a solution. This completes the proof.

7. Note first that $u_1 = 1 - u$. Since for all $x \in [u, 1]$, $u \leq x$ and $1 - x \leq 1 - u$ we have

$1 - (\sqrt{ux} + \sqrt{(1-u)(1-x)})^2$

$= 1 - ux - (1-u)(1-x) - 2\sqrt{ux(1-u)(1-x)}$

$= u + x - 2ux - 2\sqrt{ux(1-u)(1-x)}$

$\leq u + x - 2ux - 2u(1 - x) = x - u$.

Therefore, $f(x) = 0$ if $0 \leq x \leq u$ ...(i)

and

$f(x) \leq x - u$ if $u \leq x \leq 1$. ...(ii)

From (ii) we get $u_2 = f(u_1) - u = 1 - 2u$ if $u_1 \geq u$. An easy induction then yields $u_{n+1} = f(u_n) \leq u_n - u \leq 1 - (n + 1)u$ if $u_i \geq u$ for all $i = 1, 2, \ldots, n$.

Thus for sufficiently large k, we must have $u_{k-1} < u$ and then $u_k = f(u_{k-1}) 0$ by (ii).

Perhaps you have seen the magic square

8 1 6
3 5 7
4 9 2

which has the property that all rows, columns and diagonals sum to 15. Well, it has another "magic" and "square" property! If you read the rows as NUMBERS, forwards and backwards, and square them, then

$816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2$

MATHS MUSING

SOLUTION SET-188

1. (a) : The number of 5-digit numbers

$= 6 \times 6 \times 5 \times 4 \times 3 = 2160$

The possible ways of choosing 5 digits for getting number divisible by 3 are (1, 3, 4, 6, 7), (0, 1, 2, 3, 6), (0, 1, 3, 4, 7), (0, 1, 4, 6, 7), (0, 2, 3, 6, 7), (0, 2, 3, 4, 6)

The number of numbers using them = 5! + 5 (5! – 4!)

$= 120 + 5 \times 96 = \mathbf{600}$

$\therefore$ Required probability $= \dfrac{600}{2160} = \dfrac{\mathbf{5}}{\mathbf{18}}$

2. (b) : $p + 1 = 1 + \dfrac{1}{3} + \dfrac{1\cdot 3}{3\cdot 6} + \dfrac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9} + \ldots \infty$

Let $p + 1 = (1 + x)^n$

$(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{1\cdot 2}x^2 + \ldots \infty$

On comparing the coefficients of different powers of x, we get

$nx = \dfrac{1}{3} \Rightarrow n^2x^2 = \dfrac{1}{9}$ ...(i)

$n(n-1)\cdot\dfrac{x^2}{2} = \dfrac{1}{6}$ ...(ii)

Using (i) and (ii), we get

$\dfrac{2n}{(n-1)} = \dfrac{2}{3}$

$\Rightarrow 3n = n - 1 \Rightarrow n = \dfrac{-1}{2}$

So, $x = \dfrac{-2}{3}$

$\therefore (1+x)^n = \left(1 - \dfrac{2}{3}\right)^{-1/2} = \left(\dfrac{1}{3}\right)^{-1/2} = \sqrt{3}$

Hence, $(p + 1)^2 = 3$ or $p^2 + 2p - 2 = 0$.

3. (c) : Let $I = \displaystyle\int \frac{dx}{(1+\sqrt{x})\sqrt{x}\sqrt{1-x}}$

Put $1 + \sqrt{x} = t \Rightarrow \dfrac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow I = \displaystyle\int \frac{2dt}{t\sqrt{2t - t^2}}$

Again put $t = \dfrac{1}{z} \Rightarrow dt = \dfrac{-1}{z^2}dz$

$\therefore I = 2\displaystyle\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{2}{z} - \frac{1}{z^2}}} = 2\int \frac{-dz}{\sqrt{2z-1}} = -2\sqrt{2z-1} + C$

$= -2\sqrt{\dfrac{2}{t} - 1} + C = -2\sqrt{\dfrac{2-t}{t}} + C = -2\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}} + C$

4. (b) : Let $y = f(x) \Rightarrow y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

$\Rightarrow \dfrac{y+1}{y-1} = \dfrac{2^x - 2^{-x} + 2^x + 2^{-x}}{2^x - 2^{-x} - 2^x - 2^{-x}}$

$\Rightarrow \dfrac{y+1}{y-1} = -\dfrac{2^x}{2^{-x}} = -2^{2x} \Rightarrow 2^{2x} = \dfrac{1+y}{1-y}$

$\Rightarrow 2x\log 2 = \log\dfrac{1+y}{1-y}$

$\Rightarrow x = \dfrac{1}{2}\dfrac{\log\frac{1+y}{1-y}}{\log 2} = \dfrac{1}{2}\log_2\dfrac{1+y}{1-y}$

$\therefore f^{-1}(x) = \dfrac{1}{2}\log_2\dfrac{1+x}{1-x}$

5. (b,c) : Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is $x^2 + 1 = ax + b$

$\Rightarrow x^2 - ax + (1 - b) = 0$

This has equal roots when $a^2 - 4(1 - b) = 0$

or $a^2 + 4b = 4$ ...(i)

For the lower parabola, $y = -x^2$, we have

$ax + b = -x^2 \Rightarrow x^2 + ax + b = 0$

This has equal roots when $a^2 - 4b = 0$ ...(ii)

From (i) and (ii), we get $8b = 4 \Rightarrow b = 1/2$

Adding (i) and (ii), we get $2a^2 = 4 \Rightarrow a = \pm\sqrt{2}$

$\therefore$ The tangent lines are $y = \sqrt{2}x + \dfrac{1}{2}$ and $y = -\sqrt{2}x + \dfrac{1}{2}$.

6. (a, b, c) : $\vec{a} = p(\vec{x} \times (\vec{y} \times \vec{z}))$, where $p \in R - \{0\}$

$= p\{(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}\}$

$= p(\vec{y} - \vec{z})$ $\left[\because \vec{x}\cdot\vec{z} = \vec{x}\cdot\vec{y} = (\sqrt{2}\cdot\sqrt{2})\dfrac{1}{2} = 1\right]$

Now, $\vec{a}\cdot\vec{y} = p\{\vec{y}\cdot\vec{y} - \vec{y}\cdot\vec{z}\} = p\{2 - 1\} = p$

$\therefore p = \vec{a}\cdot\vec{y}$

Thus, $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{y} - \vec{z})$ ...(i)

Similarly, $\vec{b} = q(\vec{y} \times (\vec{z} \times \vec{x})) = q\{(\vec{y}\cdot\vec{x})\vec{z} - (\vec{y}\cdot\vec{z})\vec{x}\} = q(\vec{z} - \vec{x})$

Now, $\vec{b}\cdot\vec{z} = q(\vec{z}\cdot\vec{z} - \vec{z}\cdot\vec{x}) = q(2 - 1) = q$

Thus, $\vec{b} = (\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$ ...(ii)

Now, $\vec{a}\cdot\vec{b} = (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})\}$ (from (i) and (ii))

$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})(\vec{y}\cdot\vec{z}-\vec{y}\cdot\vec{x}-\vec{z}\cdot\vec{z}+\vec{z}\cdot\vec{x})$

$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})(1-1-2+1) = -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$

7. (a) : Let the line be $x = -1 + r\cos\theta$

and $y = 1 + r\sin\theta$

$y^2 = 4x \Rightarrow r^2\sin^2\theta - 2(2\cos\theta - \sin\theta)r + 5 = 0.$

If r_1, r_2 are the roots, then

$$r_1 + r_2 = \frac{2(2\cos\theta - \sin\theta)}{\sin^2\theta} \text{ and } r_1 r_2 = \frac{5}{\sin^2\theta},$$

Now, $r = \dfrac{2r_1r_2}{r_1+r_2} = \dfrac{5}{2\cos\theta - \sin\theta}$

$\Rightarrow$ $2r\cos\theta - r\sin\theta = 5$

$\therefore$ The locus of P is $2(x + 1) - (y - 1) = 5$

or $2x - y - 2 = 0.$

8. (d) : $r = \dfrac{r_1+r_2}{2} = \dfrac{2\cos\theta - \sin\theta}{\sin^2\theta}$

$\Rightarrow$ $(r\sin\theta)^2 = 2r\cos\theta - r\sin\theta$

$\Rightarrow$ $(y - 1)^2 = 2(x + 1) - (y - 1)$

$\Rightarrow$ $y^2 - 2x - y - 2 = 0.$

9. (3) : Let $A \equiv (3, 4), B \equiv (4, 3), C \equiv (1, 2)$

$\Rightarrow$ $a = \sqrt{10}, b = 2\sqrt{2}, c = \sqrt{2}, I = (x, y)$

$$x = \frac{3\sqrt{10}+8\sqrt{2}+\sqrt{2}}{\sqrt{10}+2\sqrt{2}+\sqrt{2}} = \frac{3\sqrt{10}+9\sqrt{2}}{\sqrt{10}+3\sqrt{2}} = 3.$$

10. (c) : **(P) → (1), (Q) → (2), (R) → (3), (S) → (4)**

Given, $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ is orthogonal $\Rightarrow$ $AA^T = I$

or $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or $\begin{bmatrix} p & q & q \\ q & p & q \\ q & q & p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where $p = a^2 + b^2 + c^2$ and $q = ab + bc + ca$

$\Rightarrow$ $p = a^2 + b^2 + c^2 = 1$ and $q = ab + bc + ca = 0$

$\Rightarrow$ $(a + b + c)^2 = 1$ $\Rightarrow$ $a + b + c = \pm 1$

Now, $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = \pm 1$

$\Rightarrow$ $a^3 + b^3 + c^3 = \pm 1 + 3 = 4$ or 2

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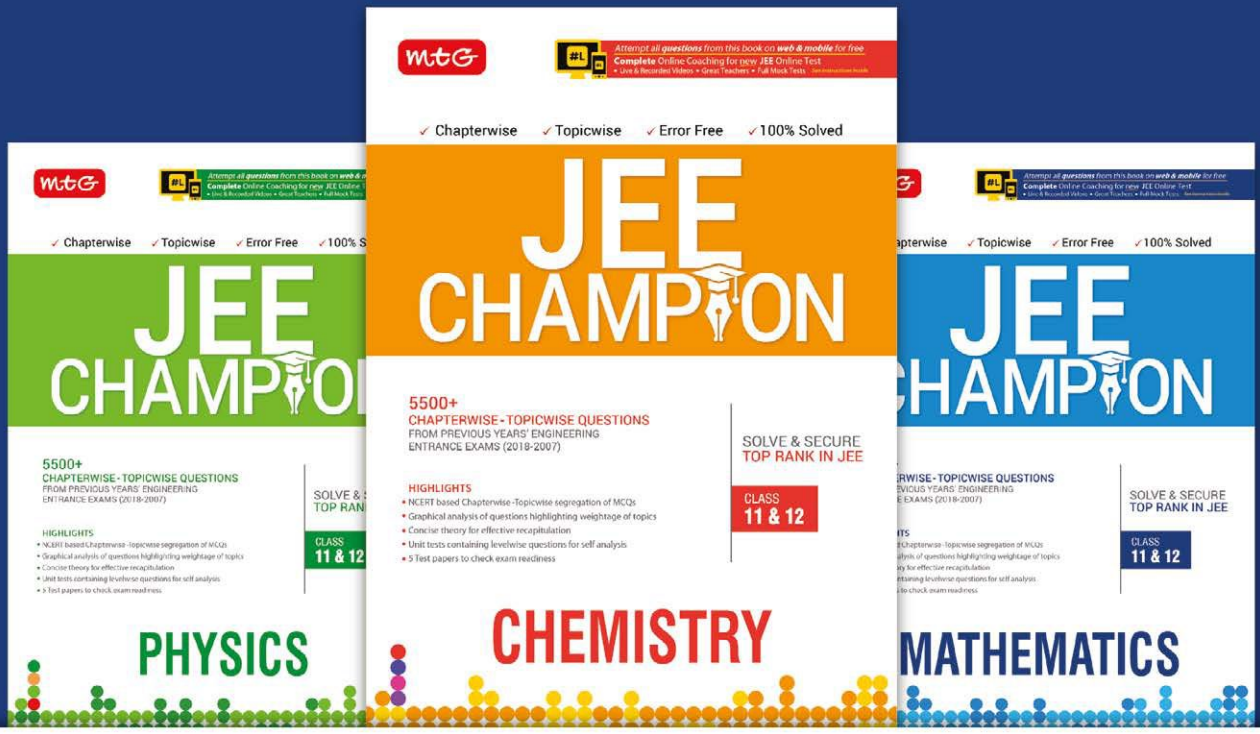
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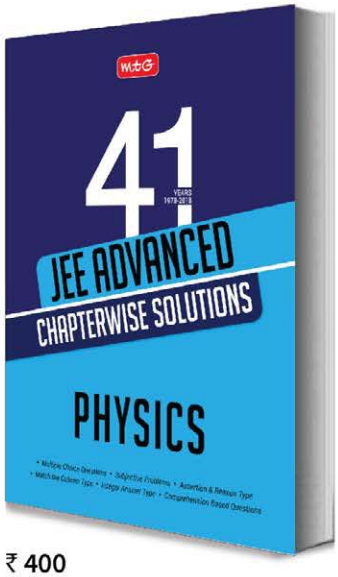

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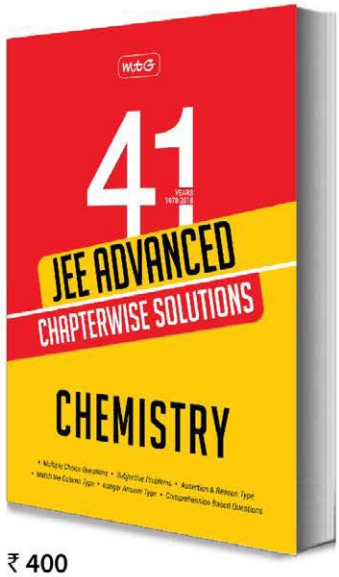

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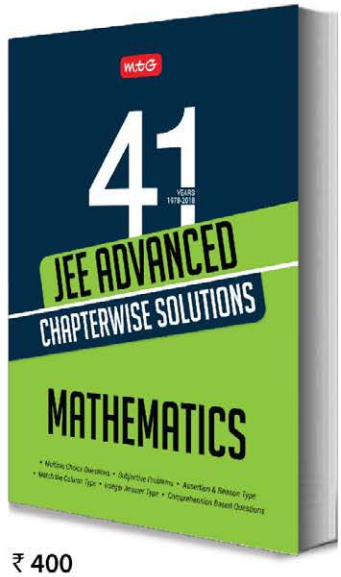

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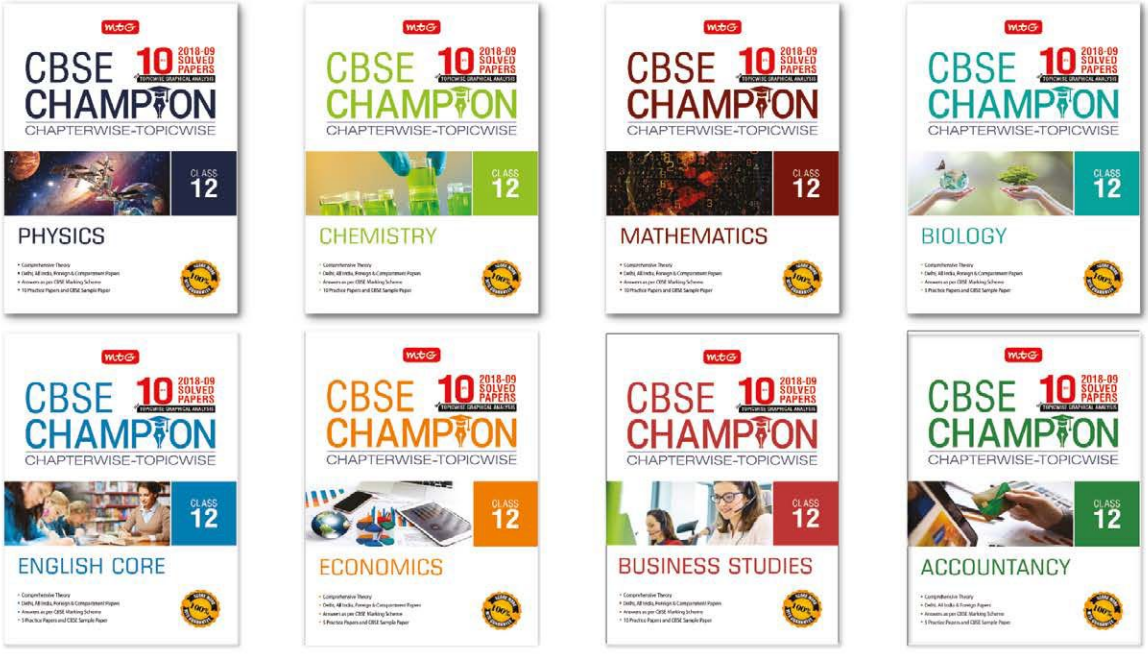

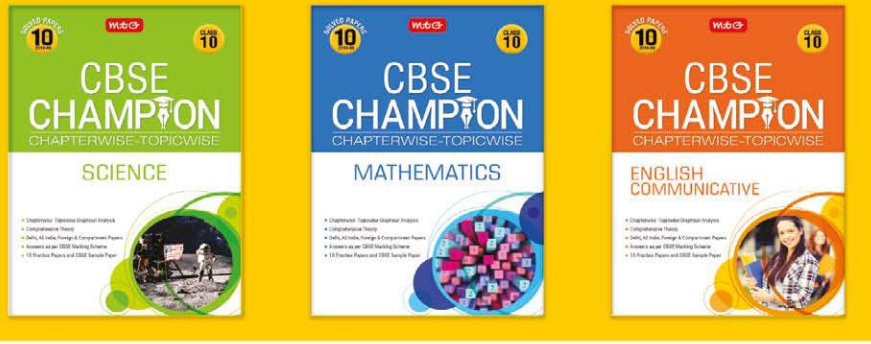

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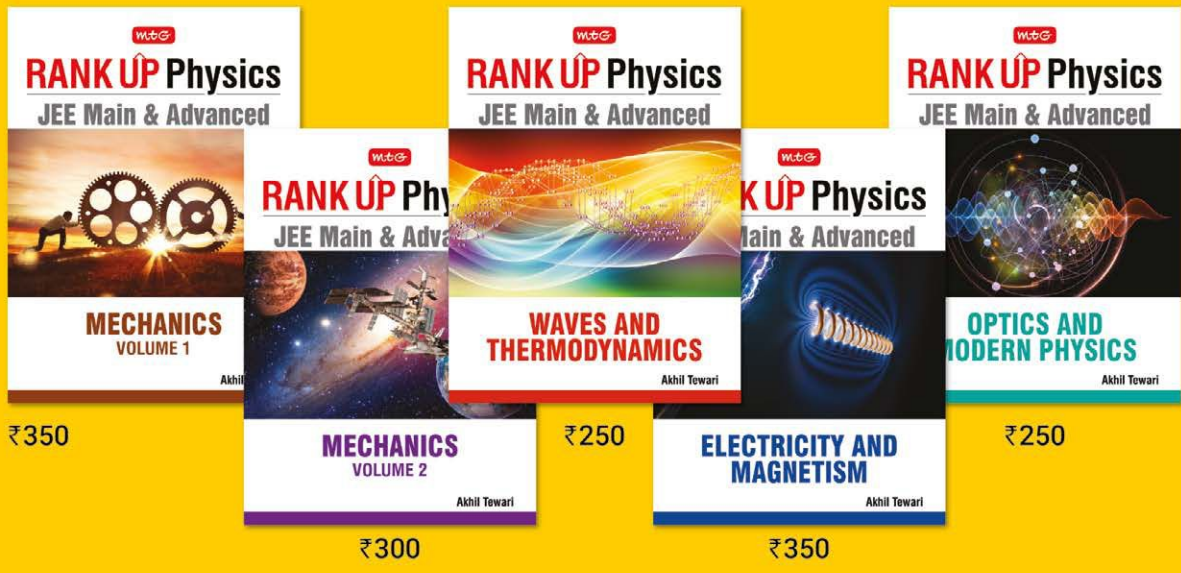

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